

HEAT CONTENT WITH SINGULAR INITIAL TEMPERATURE AND SINGULAR SPECIFIC HEAT

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ABSTRACT. Let (M, g) be a compact Riemannian manifold without boundary. Let Ω be a compact subdomain of M with smooth boundary. We examine the heat content asymptotics for the heat flow from Ω into M where both the initial temperature and the specific heat are permitted to have controlled singularities on $\partial\Omega$. The operator driving the heat process is assumed to be an operator of Laplace type.

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1. INTRODUCTION

The conduction of heat or the diffusion of matter through the parts of a body is a classical subject in engineering. For an early reference we refer to the work of Carslaw and Jaeger [9], and the references therein. From a mathematical point of view both heat content and heat trace link the spectral resolution of an operator of Laplace type to the underlying geometry of the manifold and its boundary in a natural way. The heat trace shows up when calculating the thermodynamic properties of quantum systems. See e.g. [1]. A considerable amount of progress has been made in the last few decades in the understanding of the short time asymptotic behaviour of the heat content in a variety of settings and boundary conditions [12]. It was discovered by Preunkert [13, 14, 16] that if we consider the setting where a region Ω in Euclidean space \mathbb{R}^m is at initial temperature 1 while the complement is at initial temperature 0 then the integral of the temperature in Ω at time t denoted by $Q_\Omega(t)$ satisfies

$$Q_\Omega(t) = |\Omega| - \pi^{-1/2} \mathcal{P}(\Omega) t^{1/2} + o(t), \quad t \downarrow 0,$$

where $|\Omega|$ denotes the measure of Ω , and $\mathcal{P}(\Omega)$ denotes the perimeter of Ω . Apparently $\mathbb{R}^m \setminus \Omega$ acts for a very small time approximately as a 0 Dirichlet boundary condition. It was subsequently shown [3, 4] that many of the asymptotic properties of $Q_\Omega(t) \downarrow 0$ are similar to the situation where Dirichlet or Neumann boundary conditions on the boundary of Ω have been imposed [12]. This paper investigates properties of heat flow and heat content in the setting of singular specific heat, and singular initial temperature distributions. The latter were first examined in [2, 8, 6] where either the specific heat or initial boundary condition were singular but not both. It turns out that in the absence of boundary conditions but in the presence of doubly singular data a suitable asymptotic series exist for $t \downarrow 0$ but that the corresponding existence proof is not as straightforward as one might expect. Previously, in the presence of both boundary conditions and doubly singular data, the existence was part of the hypothesis [7]. This paper is a first step towards a proof of an asymptotic series in the doubly singular setting with boundary conditions. We also note that while the same collection of local geometric invariants appear as coefficients in the series, the numerical coefficients have to be computed by a new collection of special case calculations such as the interval in \mathbb{R} .

Let (M, g) be a compact smooth Riemannian manifold of dimension m without boundary. Let D_M be an operator of Laplace type on a smooth vector bundle V over M . This means that

$$D_M = - \{ g^{ij}(x) \partial_{x_i} \partial_{x_j} + a_1^i(x) \partial_{x_i} + a_0(x) \} \quad (1.a)$$

in a system of local coordinates (x^1, \dots, x^m) and relative to some local frame for V . We adopt the *Einstein convention* and sum over repeated indices. Let Ω be a subdomain of M , i.e. Ω is an m -dimensional submanifold of M . We assume that Ω is compact with smooth boundary $\partial\Omega$. Let

$\phi \in L^1(V|_\Omega)$ represent the initial temperature and let $\rho \in L^1(V^*|_\Omega)$ represent the specific heat. We let ϕ_Ω and ρ_Ω be the extension of ϕ and ρ to M to be zero on the complement of Ω . The heat content of Ω in M is given for $t > 0$ by:

$$\beta_\Omega(\phi, \rho, D_M)(t) := \int_M \langle e^{-tD_M} \phi_\Omega, \rho_\Omega \rangle dx$$

where $\langle \cdot, \cdot \rangle$ denotes the natural pairing between V and the dual bundle V^* , where e^{-tD_M} is the fundamental solution of the heat equation for the operator D_M , and where dx is the Riemannian volume element of M . If $K_M(x, \tilde{x}; t)$ is the kernel of e^{-tD_M} , then:

$$\beta_\Omega(\phi, \rho, D_M)(t) = \int_\Omega \int_\Omega \langle K_M(x, \tilde{x}; t) \phi(x), \rho(\tilde{x}) \rangle dx d\tilde{x}.$$

This quantity has been studied previously where the ambient manifold M was Euclidean space [3] and where both ϕ and ρ were equal to 1 on Ω under the very mild condition that the characteristic function of Ω was of bounded variation. There it was also shown that the heat content of Ω in M could be finite for all $t > 0$ even if Ω has infinite Lebesgue measure. The situation described below is similar in spirit.

1.1. Growth assumptions on the initial temperature and on the specific heat. Let

$$\mathcal{O} := \{(a, b) \in \mathbb{C} : \Re(a) < 1, \Re(b) < 1, a + b \neq 1, -1, -3, \dots\}. \quad (1.b)$$

Let ϕ and ρ be smooth on the interior of Ω . Let $r(x)$ be the geodesic distance from a point x in Ω to the boundary $\partial\Omega$. The function r is not smooth on all of Ω . However r is smooth if restricted to a small open neighborhood \mathcal{A} of $\partial\Omega$ in Ω . Assume in addition that $r^a \phi \in C^\infty(V|_\mathcal{A})$ and that $r^b \rho \in C^\infty(V^*|_\mathcal{A})$. In this setting, we say that $r^a \phi$ and $r^b \rho$ are *smooth near the boundary*. Although in practice, one is only interested in real (a, b) , we shall see that it is convenient to pass to the complex setting to be able to use analytic continuation. Since $\Re(a) < 1$ and $\Re(b) < 1$, ϕ and ρ are in L^1 . If $\Re(b) > 0$, then ρ has a controlled blowup while if $\Re(b) < 0$, then ρ has a controlled decay as we approach $\partial\Omega$ from within Ω . If $b = 0, -1, -2, \dots$, then in fact ρ is smooth on all of Ω when Ω is regarded as a compact manifold with boundary in its own right. Viewed as a function on all of M , of course, ρ_Ω is not smooth on $\partial\Omega$ since ρ_Ω is zero on Ω^c . If k is a non-negative integer with $\Re(b) < -k$, then ρ_Ω is C^k on M . The situation is similar for ϕ .

1.2. The asymptotic series for the heat content. The following is one of the two main results of this paper:

Theorem 1.1. *Let D_M be an operator of Laplace type on a smooth vector bundle V over a compact Riemannian manifold (M, g) without boundary. Let Ω be a compact subdomain of M with smooth boundary. Let $\phi \in C^\infty(V|_{\text{int}(\Omega)})$ and let $\rho \in C^\infty(V^*|_{\text{int}(\Omega)})$. Let $(a, b) \in \mathcal{O}$. We assume that $r^a \phi$ and $r^b \rho$ are smooth near the boundary of Ω . Let $\beta_\Omega(\phi, \rho, D_M)(t)$ be the heat content of Ω in M . Then there is a complete asymptotic expansion of $\beta_\Omega(\phi, \rho, D_M)(t)$ for small time such that for any positive integer N as $t \downarrow 0$ we have:*

$$\beta_\Omega(\phi, \rho, D_M)(t) = \sum_{n=0}^N t^n \beta_{n,a,b}^\Omega(\phi, \rho, D_M) + \sum_{j=0}^N t^{(1+j-a-b)/2} \beta_{j,a,b}^{\partial\Omega}(\phi, \rho, D_M) + O(t^{(N-1)/2}).$$

The coefficient $\beta_{0,a,b}^{\partial\Omega}(\phi, \rho, D_M)$ of $t^{(1+j-a-b)/2}$ is given by

$$\begin{aligned} \beta_{0,a,b}^{\partial\Omega}(\phi, \rho, D_M) &= c(a, b) \int_{\partial\Omega} \langle \phi_0, \rho_0 \rangle dy, \quad \text{where} \\ c(a, b) &:= 2^{-a-b} \frac{1}{\sqrt{\pi}} \Gamma\left(\frac{2-a-b}{2}\right) \Gamma(a+b-1) \cdot \left(\frac{\Gamma(1-a)}{\Gamma(b)} + \frac{\Gamma(1-b)}{\Gamma(a)} \right). \end{aligned}$$

More generally, the coefficients $\beta_{n,a,b}^\Omega(\cdot)$ are given as regularized integrals of local invariants over the interior of Ω that are bilinear in the derivatives of $\{\phi, \rho\}$ up to order $2n$ with coefficients that depend holomorphically on the parameters (a, b) , that depend smoothly on the 0-jets of the metric,

and that are polynomial in the derivatives of the total symbol of D_M up to order $2n$. The coefficients $\beta_j^{\partial\Omega}(\cdot)$ are given similarly as integrals of local invariants over the boundary $\partial\Omega$ where the derivatives of $\{\phi, \rho\}$ and of the total symbol of D_M are up to order j .

If the boundary of Ω is not connected, then we can allow different values of (a, b) (i.e. different rates of growth or decay) on the different components of $\partial\Omega$ and there are separate asymptotic series arising from each component of the boundary.

1.3. Log terms in the asymptotic expansion. The assumption that $a + b \neq 1, -1, -3, \dots$ in the definition of \mathcal{O} which was given in Equation (1.b) is essential in Theorem 1.1; there can be logarithmic singularities when $a + b = 1, -1, -3, \dots$. Let $K_{\mathbb{R}}(x, \tilde{x}; t)$ be the heat kernel of $D_{\mathbb{R}} := -\partial_x^2$ on \mathbb{R} :

$$K_{\mathbb{R}}(x, \tilde{x}; t) := \frac{1}{\sqrt{4\pi t}} e^{-(x-\tilde{x})^2/(4t)}. \quad (1.c)$$

If ρ and ϕ belong to $L^1(\Omega)$, then the heat content of $[0, 1]$ in \mathbb{R} is given by:

$$\beta_{[0,1]}(\phi, \rho, D_{\mathbb{R}})(t) := \int_0^1 \int_0^1 K_{\mathbb{R}}(x, \tilde{x}; t) \phi(x) \rho(\tilde{x}) x^{-a} \tilde{x}^{-b} dx d\tilde{x}.$$

The following is the second main result of this paper:

Theorem 1.2. *Let $(a, b) \in \mathbb{R}^2$ with $a < 1$ and $b < 1$. Assume that $a + b = 1$. Let Ξ_i be smooth monotonically decreasing cut-off functions which are identically 1 near $x = 0$ and identically 0 in an open neighborhood of the interval $[1/2, 1]$. Then for $t \downarrow 0$*

$$\beta_{[0,1]}(x^{-a}\Xi_1, x^{-b}\Xi_2, D_{\mathbb{R}})(t) = \beta_0(a, b, \Xi_1, \Xi_2) - \frac{1}{2} \log(t) + O(t^{\frac{1}{2}} \log(t))$$

where for any $\epsilon > 0$ sufficiently small,

$$\begin{aligned} \beta_0(a, b, \Xi_1, \Xi_2) &= \frac{1}{2} \log(\epsilon^2) + \frac{1}{2} \gamma + \log(2^{1/2} - 1) + 2 \log 2 + \int_{[\epsilon, \frac{1}{2}]} \Xi_1(x) \Xi_2(x) x^{-1} dx \\ &\quad + \frac{1}{2} \int_{[0,1]} dq q^{-1} \left(\frac{(1+q)^{a-1}}{(1-q)^a} + \frac{(1-q)^{a-1}}{(1+q)^a} - \frac{2}{(1+q^2)^{1/2}} \right). \end{aligned}$$

The parameter ϵ serves to regularize the integral and does not contribute to the value of β_0 . We refer to Theorem 4.4 for additional information concerning the log terms in a more general context.

1.4. A more general setting. Let D_M be the Laplace-Beltrami operator. We have defined the heat content of Ω in M if the ambient manifold M is a compact smooth Riemannian manifold without boundary. But it can also be defined using the Dirichlet realization of D_M if M is a smooth compact Riemannian manifold with boundary. And it can be defined if M is a complete Riemannian manifold such that the Ricci curvature of M is bounded from below. Theorem 1.3 below was proved in [4]. It will let us pass between regarding Ω as a subdomain of a compact manifold without boundary, as a compact subdomain of a manifold with boundary, or a compact subdomain of a complete manifold under very mild conditions. Let $\Omega \subset \text{Interior}\{\tilde{M}\} \subset \tilde{M} \subset M$ where Ω and \tilde{M} are compact subdomains with smooth boundaries of a Riemannian manifold (M, g) . Let $\epsilon := \text{dist}_g(\partial\Omega, \partial\tilde{M}) > 0$. Let Δ_M be the Laplace-Beltrami operator on M . Let $\Delta_{\tilde{M}}$ be the Dirichlet realization of D_M on \tilde{M} .

Theorem 1.3. *Assume that (M, g) is complete with non-negative Ricci curvature. Let ρ be continuous on M and let $\phi \in L^1(\Omega)$. Then:*

$$|\beta_{\Omega}(\phi, \rho, \Delta_{\tilde{M}})(t) - \beta_{\Omega}(\phi, \rho, \Delta_M)(t)| \leq 2^{(2+m)/2} \|\phi\|_{L^1(\Omega)} \|\rho\|_{L^\infty(\Omega)} e^{-\epsilon^2/(8t)}.$$

1.5. Previous results. It is worth presenting an example to illustrate the kinds of coefficients which arise in the asymptotic series of Theorem 1.1. We first introduce some additional notation. If D_M is an operator of Laplace type, then there is a unique connection ∇ on V and a unique endomorphism E of V so that we can express D_M in a Bochner formalism:

$$D_M = -(g^{ij} \nabla_{\partial x_i} \nabla_{\partial x_j} + E).$$

Near the boundary, we introduce local coordinates (r, y) so that the curves $\sigma_t : t \rightarrow (t, y)$ are unit speed geodesics which are perpendicular to the boundary when $t = 0$. Assume that $b = 0$ so ρ is smooth on Ω . Near the boundary, we may expand

$$\phi(r, y) \sim \sum_{i=0}^{\infty} r^{i-a} \phi_i(y) \text{ and } \rho(r, y) \sim \sum_{j=0}^{\infty} r^j \phi_j$$

in modified Taylor series. We use the connection ∇ on V and the dual connection ∇^* on V^* to introduce local frames for V and V^* which are parallel along the geodesics σ_t . Let L_{uv} be the components of the second fundamental form relative to a local orthonormal frame for the tangent bundle of the boundary, let Ric_{mm} be the Ricci curvature of the normal vector field, and let τ be the scalar curvature. Let “ $;$ ” denote the components of covariant differentiation with respect to ∇ on V and with respect to ∇^* on V^* . The following result [4] will be an essential ingredient in the proofs that we shall give of Theorem 1.1 and of Theorem 1.2 subsequently:

Theorem 1.4. *Adopt the notation of Theorem 1.1. Let $b = 0$. There is a complete asymptotic series as $t \downarrow 0$ of the form:*

$$\beta_{\Omega}(\phi, \rho, D_M)(t) \sim \sum_{n=0}^{\infty} t^n \beta_n^{\Omega}(\phi, \rho, D_M) + \sum_{j=0}^{\infty} t^{(1+j-a)/2} \beta_{j,a}^{\partial\Omega}(\phi, \rho, D_M),$$

where β_n^{Ω} and $\beta_{j,a}^{\partial\Omega}$ are integrals of certain locally computable invariants over Ω and $\partial\Omega$, respectively, and where the dependence on (a, b) is holomorphic. In particular, we have:

$$\begin{aligned} \beta_{0,a}^{\partial\Omega}(\phi, \rho, D_M) &= 2c(a, 0) \int_{\partial\Omega} \frac{1}{2} \langle \phi_0, \rho_0 \rangle dy, \\ \beta_{1,a}^{\partial\Omega}(\phi, \rho, D_M) &= 2c(a-1, 0) \int_{\partial\Omega} \left\{ \frac{1}{2} \langle \phi_1, \rho_0 \rangle + \frac{a}{4(1-a)} \langle L_{uu} \phi_0, \rho_0 \rangle + \frac{1}{2(a-1)} \langle \phi_0, \rho_1 \rangle \right\} dy, \\ \beta_{2,a}^{\partial\Omega}(\phi, \rho, D_M) &= 2c(a-2, 0) \int_{\partial\Omega} \left\{ \frac{1}{2} \langle \phi_2, \rho_0 \rangle + \frac{a-1}{4(2-a)} \langle L_{uu} \phi_1, \rho_0 \rangle + \frac{3-a}{4(1-a)(2-a)} \langle E \phi_0, \rho_0 \rangle \right. \\ &\quad + \frac{1}{(1-a)(2-a)} \langle \phi_0, \rho_2 \rangle - \frac{a+1}{4(2-a)(1-a)} \langle L_{uu} \phi_0, \rho_1 \rangle - \frac{1-a}{8(2-a)} \langle \text{Ric}_{mm} \phi_0, \rho_0 \rangle \\ &\quad + \frac{3a-5}{16(2-a)} \langle L_{uu} L_{vv} \phi_0, \rho_0 \rangle - \frac{1-a}{8(2-a)} \langle L_{uv} L_{uv} \phi_0, \rho_0 \rangle \\ &\quad \left. - \frac{3-a}{4(1-a)(2-a)} \langle \phi_{0;u}, \rho_{0;u} \rangle + 0 \langle \tau \phi_0, \rho_0 \rangle + \frac{1}{2(a-2)} \langle \phi_1, \rho_1 \rangle \right\} dy. \end{aligned}$$

Remark 1.5. We note that if $b = -1, -2, \dots$, then ρ is smooth on Ω and this result continues to hold. Since $\rho_j = 0$ for $j = 0, \dots, -b+1$ in this setting, then we can re-index the series for the boundary invariants to involve powers $t^{(1+j-a-b)/2}$. This establishes Theorem 1.1 if $b = 0, -1, -2, \dots$; again if the boundary of Ω contains several components, we can permit (a, b) to vary with the component and add the resulting boundary contributions.

1.6. Outline of the paper. In Section 2, we shall establish Theorem 1.2. The proof of Theorem 1.1 is considerably more lengthy. Let $(a, b) \in \mathcal{O}$ where \mathcal{O} is defined in Equation 1.b. In Section 3, we establish the existence of an asymptotic series of the form given in Theorem 1.1 for the special case that $\Omega = [0, 1]$ is embedded in \mathbb{R} , that $\phi(x) = x^{-a}$, and that $\rho(x) = x^{-b}$; see Theorem 3.1 for details. In Section 4, we use the pseudo-differential calculus depending upon a complex parameter which was developed by Seeley [17, 18] to complete the proof of Theorem 1.1; the special case computation of Theorem 3.1 being an essential input. We shall use the treatment of the pseudo-differential calculus given in [11]. Our discussion follows closely that of [4] although there are some important differences. In Section 4.5 (see Theorem 4.4) we generalize Theorem 1.1 and Theorem 1.2 to establish the existence of a complete asymptotic series with log terms when $a + b = 1 - 2k$, $a < 1$, $b < 1$, and $(a, b) \in \mathbb{R}^2$. We conclude the paper in Section 5 by using the methods of this paper to study the heat content with Neumann and Dirichlet boundary conditions for the operator $-\partial_x^2$ on the interval.

2. THE PROOF OF THEOREM 1.2 – LOG TERMS

To prove Theorem 1.2, we assume that $a + b = 1$ and adapt an argument of [5] to obtain a precise formula for the first 2 terms in the asymptotic series. Let $(a, b) \in \mathbb{R}^2$ with $a < 1$, $b < 1$, and $a + b = 1$. We choose the notation so $0 < b \leq a < 1$. We shall apply an argument from [5]. Let $0 < \epsilon_1 < \epsilon_3 < 1/2$, and let $0 < \epsilon_2 < \epsilon_4 < 1/2$. We assume that Ξ_1 (resp. Ξ_2) is a smooth monotonically decreasing function on $[0, 1]$ which is identically 1 for $0 \leq x \leq \epsilon_1$ (resp. $0 \leq x \leq \epsilon_2$) and which vanish identically for $\epsilon_3 \leq x$ (resp. $\epsilon_4 \leq x$). Let

$$\begin{aligned} C &= \{(x, \tilde{x}) \in \mathbb{R} : x^2 + \tilde{x}^2 \geq \epsilon^2, 0 \leq x \leq \epsilon_3, 0 \leq \tilde{x} \leq \epsilon_4\}, \\ C_1 &= \{(x, \tilde{x}) \in C : |x - \tilde{x}| \leq \sigma\}, \end{aligned}$$

where $\sigma \in (0, \epsilon/5)$ will be chosen later on. We decompose the heat content as follows.

$$\begin{aligned} \beta_{[0,1]}(x^{-a}\Xi_1, x^{-b}\Xi_2, D_{\mathbb{R}})(t) &= B_1(t) + B_2(t), \text{ where} \\ B_1(t) &:= \iint_{\mathbb{R}_+^2 \cap \{x^2 + \tilde{x}^2 < \epsilon^2\}} K_{\mathbb{R}}(x, \tilde{x}; t) x^{-a} \tilde{x}^{-b} dx d\tilde{x}, \\ B_2(t) &:= \iint_C K_{\mathbb{R}}(x, \tilde{x}; t) \Xi_1(x) \Xi_2(\tilde{x}) x^{-a} \tilde{x}^{-b} dx d\tilde{x}. \end{aligned} \tag{2.a}$$

To estimate $B_2(t)$ we first consider the contribution from the set $C \setminus C_1$. We have that

$$K_{\mathbb{R}}(x, \tilde{x}; t) \leq t^{-1/2} e^{-(x-\tilde{x})^2/(4t)} \leq t^{-1/2} e^{-\sigma^2/(4t)}, (x, \tilde{x}) \in C \setminus C_1.$$

Consequently

$$\begin{aligned} \iint_{C \setminus C_1} K_{\mathbb{R}}(x, \tilde{x}; t) \Xi_1(x) \Xi_2(\tilde{x}) x^{-a} \tilde{x}^{-b} dx d\tilde{x} \\ \leq K t^{-1/2} e^{-\sigma^2/(4t)}, \text{ where } K = \iint_C \Xi_1(x) \Xi_2(\tilde{x}) x^{-a} \tilde{x}^{-b} dx d\tilde{x}. \end{aligned} \tag{2.b}$$

On the set $C \cap \{|x - \tilde{x}| \leq \epsilon/5\}$ we have that $\tilde{x} \rightarrow \Xi_2(\tilde{x}) \tilde{x}^{-b}$ is C^∞ . Hence there exists $L = L(\epsilon, b, \Xi_2)$ such that $|\Xi_2(\tilde{x}) \tilde{x}^{-b} - \Xi_2(x) x^{-b}| \leq L|x - \tilde{x}|$. Because $x \geq \epsilon/2$ and $\tilde{x} \geq \epsilon/2$ on $C \cap \{|x - \tilde{x}| \leq \epsilon/5\}$ and because $K_{\mathbb{R}}(x, \tilde{x}; t) \leq t^{-1/2}$, we have that

$$\begin{aligned} \iint_{C_1} K_{\mathbb{R}}(x, \tilde{x}; t) \Xi_1(x) x^{-a} L|x - \tilde{x}| dx d\tilde{x} &\leq L(2/\epsilon)^a t^{-1/2} \iint_{C_1} |x - \tilde{x}| dx d\tilde{x} \\ &\leq 2L(2/\epsilon)^a t^{-1/2} \sigma^2. \end{aligned} \tag{2.c}$$

We now choose σ^2 as to minimize $t^{-1/2} e^{-\sigma^2/(4t)} + t^{-1/2} \sigma^2$ by taking:

$$\sigma^2 = 4t \log \left(\frac{1}{4t} \right).$$

This gives that for t sufficiently small, the right hand sides of Equation (2.b) and of Equation (2.c) are $O(t^{1/2})$ and $O(t^{1/2} \log(t^{-1}))$, respectively. We conclude that

$$B_2(t) = \iint_{C_1} K_{\mathbb{R}}(x, \tilde{x}; t) \Xi_1(x) \Xi_2(x) x^{-1} dx d\tilde{x} + O(t^{1/2} \log(t^{-1})). \quad (2.d)$$

We now write

$$C_1 = (C_1 \cap \{x^2 \geq \epsilon^2/2\}) \cup (C_1 \cap \{x^2 < \epsilon^2/2\}) = C_2 \cup C_3.$$

Since $x \geq \epsilon/2$ on C_1 , the integrand in the first term in the right hand side of Equation (2.d) is bounded by $2\epsilon^{-1}t^{-1/2}$. Hence

$$\iint_{C_3} K_{\mathbb{R}}(x, \tilde{x}; t) \Xi_1(x) \Xi_2(x) x^{-1} dx d\tilde{x} \leq 2\epsilon^{-1}t^{-1/2}|C_3|,$$

where $|\cdot|$ denotes Lebesgue measure. Because $|C_3| \leq \sigma^2/2$,

$$0 \leq \iint_{C_3} K_{\mathbb{R}}(x, \tilde{x}; t) \Xi_1(x) \Xi_2(x) x^{-1} dx d\tilde{x} \leq \epsilon^{-1}t^{-1/2}\sigma^2,$$

and so the contribution from C_3 to the integral in Equation (2.d) is $O(t^{1/2} \log(t^{-1}))$. Furthermore

$$\begin{aligned} & \iint_{C_2} K_{\mathbb{R}}(x, \tilde{x}; t) \Xi_1(x) \Xi_2(x) x^{-1} dx d\tilde{x} \\ & \leq \iint_{\{x^2 \geq \epsilon^2/2\}} K_{\mathbb{R}}(x, \tilde{x}; t) \Xi_1(x) \Xi_2(x) x^{-1} dx d\tilde{x} \\ & = \int_{[\epsilon/\sqrt{2}, 1/2]} \Xi_1(x) \Xi_2(x) x^{-1} dx. \end{aligned}$$

To obtain a lower bound for the contribution from C_2 to the integral in Equation (2.d), we note that

$$\begin{aligned} & \iint_{C_2} K_{\mathbb{R}}(x, \tilde{x}; t) \Xi_1(x) \Xi_2(x) x^{-1} dx d\tilde{x} \\ & \geq \iint_{\{x^2 \geq \epsilon^2/2\}} K_{\mathbb{R}}(x, \tilde{x}; t) \Xi_1(x) \Xi_2(x) x^{-1} dx d\tilde{x} \\ & \quad - \iint_{\{|x-\tilde{x}| \geq \sigma\} \cap \{x^2 \geq \epsilon^2/2\}} K_{\mathbb{R}}(x, \tilde{x}; t) \Xi_1(x) \Xi_2(x) x^{-1} dx d\tilde{x} \\ & = \int_{[\epsilon/\sqrt{2}, 1/2]} \Xi_1(x) \Xi_2(x) x^{-1} dx \\ & \quad - \iint_{\{|x-\tilde{x}| \geq \sigma\} \cap \{x^2 \geq \epsilon^2/2\}} K_{\mathbb{R}}(x, \tilde{x}; t) \Xi_1(x) \Xi_2(x) x^{-1} dx d\tilde{x}. \end{aligned}$$

Moreover

$$\begin{aligned} \int_{\{|x-\tilde{x}| \geq \sigma\}} K_{\mathbb{R}}(x, \tilde{x}; t) d\tilde{x} & \leq \int_{\{|x-\tilde{x}| \geq \sigma\}} (4\pi t)^{-1/2} e^{-|x-\tilde{x}|^2/(4t)} d\tilde{x} \\ & = 4\pi^{-1/2} t^{1/2} \sigma^{-1} e^{-\sigma^2/(4t)} = O(t^2). \end{aligned}$$

Because $\Xi_1(x) \Xi_2(x) x^{-1} = x^{-1}$ for $0 < x \leq \epsilon$, we have:

$$\begin{aligned} B_2(t) & = \int_{[\epsilon/\sqrt{2}, 1/2]} \Xi_1(x) \Xi_2(x) x^{-1} dx + O(t^{1/2} \log(t^{-1})) \\ & = \int_{[\epsilon, 1/2]} \Xi_1(x) \Xi_2(x) x^{-1} dx + 2^{-1} \log 2 + O(t^{1/2} \log(t^{-1})), \end{aligned}$$

In order to obtain the asymptotic behaviour of B_1 in Equation (2.a), we introduce polar coordinates $x = (4t)^{1/2} \varrho \cos \theta$, $y = (4t)^{1/2} \varrho \sin \theta$ to find that

$$B_1(t) = \pi^{-1/2} \int_{[0, \pi/2]} d\theta (\cos \theta)^{-a} (\sin \theta)^{a-1} \int_{[0, \epsilon/(4t)^{1/2}]} d\varrho e^{-\varrho^2(1-\sin(2\theta))}.$$

A further change of variable $\theta = \Phi + \pi/4$ yields that

$$\begin{aligned} B_1(t) &= (2/\pi)^{1/2} \int_{[0, \pi/4]} d\Phi \left(\frac{(\cos \Phi + \sin \Phi)^{a-1}}{(\cos \Phi - \sin \Phi)^a} + \frac{(\cos \Phi - \sin \Phi)^{a-1}}{(\cos \Phi + \sin \Phi)^a} \right) \\ &\quad \times \int_{[0, \epsilon/(4t)^{1/2}]} d\varrho e^{-2\varrho^2(\sin \Phi)^2} = B_3(t) + B_4(t) + B_5(t). \end{aligned}$$

We make the change of variables $\tan \Phi = q$ to see:

$$\begin{aligned} B_3(t) &= (2/\pi)^{1/2} \int_{[0, \pi/4]} d\Phi \left(\frac{(\cos \Phi + \sin \Phi)^{a-1}}{(\cos \Phi - \sin \Phi)^a} + \frac{(\cos \Phi - \sin \Phi)^{a-1}}{(\cos \Phi + \sin \Phi)^a} - 2 \right) \int_{[0, \infty)} d\varrho e^{-2\varrho^2(\sin \Phi)^2} \\ &= 2^{-1} \int_{[0, \pi/4]} d\Phi (\sin \Phi)^{-1} \left(\frac{(\cos \Phi + \sin \Phi)^{a-1}}{(\cos \Phi - \sin \Phi)^a} + \frac{(\cos \Phi - \sin \Phi)^{a-1}}{(\cos \Phi + \sin \Phi)^a} - 2 \right) \\ &= 2^{-1} \int_{[0, 1]} dq q^{-1} \left(\frac{(1+q)^{a-1}}{(1-q)^a} + \frac{(1-q)^{a-1}}{(1+q)^a} - \frac{2}{(1+q^2)^{1/2}} \right). \end{aligned}$$

We also compute that

$$\begin{aligned} B_4(t) &= -(2/\pi)^{1/2} \int_{[0, \pi/4]} d\Phi \left(\frac{(\cos \Phi + \sin \Phi)^{a-1}}{(\cos \Phi - \sin \Phi)^a} + \frac{(\cos \Phi - \sin \Phi)^{a-1}}{(\cos \Phi + \sin \Phi)^a} - 2 \right) \\ &\quad \times \int_{[\epsilon/(4t)^{1/2}, \infty)} d\varrho e^{-2\varrho^2(\sin \Phi)^2}, \end{aligned}$$

and that

$$B_5(t) = (8/\pi)^{1/2} \int_{[0, \pi/4]} d\Phi \int_{[0, \epsilon/(4t)^{1/2}]} d\varrho e^{-2\varrho^2(\sin \Phi)^2}. \quad (2.e)$$

The expression under (2.e) equals

$$\begin{aligned} &(8/\pi)^{1/2} \int_{[0, \pi/4]} d\Phi \int_{[0, \epsilon/(4t)^{1/2}]} d\varrho e^{-2\varrho^2(\sin \Phi)^2} \\ &= (8/\pi)^{1/2} \int_{[0, \pi/4]} d\Phi \int_{[0, \epsilon/(4t)^{1/2}]} d\varrho e^{-2\varrho^2\Phi^2} \\ &\quad + \int_{[0, \pi/4]} d\Phi ((\sin \Phi)^{-1} - \Phi^{-1}) \\ &\quad + (8/\pi)^{1/2} \int_{[0, \pi/4]} d\Phi \int_{[\epsilon/(4t)^{1/2}, \infty)} d\varrho \left(e^{-2\varrho^2\Phi^2} - e^{-2\varrho^2(\sin \Phi)^2} \right). \end{aligned} \quad (2.f)$$

The third term in the right hand side of Equation (2.f) is $O(e^{-\epsilon^2/(5t)})$. The second term in the right hand side of (2.f) is equal to $\log(2^{1/2} - 1) + 3 \log 2 - \log \pi$. The first term in the right hand side of (2.f) equals

$$\begin{aligned} &(4/\pi)^{1/2} \int_{[0, \pi\epsilon/(32t)^{1/2}]} d\Phi \int_{[0, \Phi]} d\varrho e^{-\varrho^2} \\ &= (4/\pi)^{1/2} \log \left(\frac{\pi\epsilon}{\sqrt{32t}} \right) \int_{[0, \pi\epsilon/(32t)^{1/2}]} d\varrho e^{-\varrho^2} - \frac{2}{\sqrt{\pi}} \int_{[0, \pi\epsilon/(32t)^{1/2}]} d\Phi (\log \Phi) e^{-\Phi^2} \\ &= 2^{-1} \log(\epsilon^2/t) + \log \pi + 2^{-1}\gamma - 3 \cdot 2^{-1} \log 2 + O(e^{-\epsilon^2/(5t)}), \end{aligned}$$

where we have used Equation (4.333) in [15] together with

$$\int_{[0, \pi\epsilon/(32t)^{1/2}]} d\Phi (\log \Phi) e^{-\Phi^2} = \int_{[0, \infty)} d\Phi (\log \Phi) e^{-\Phi^2} + O(e^{-\epsilon^2/(5t)}).$$

We find that

$$B_5(t) = 2^{-1} \log(\epsilon^2/t) + 2^{-1}\gamma + \log(2^{1/2} - 1) + 3 \cdot 2^{-1} \log 2 + O(e^{-\epsilon^2/(5t)}).$$

In order to estimate $B_4(t)$ we first note that by expanding $\sin \Phi$ and $\cos \Phi$ around 0 we have that

$$\frac{(\cos \Phi + \sin \Phi)^{a-1}}{(\cos \Phi - \sin \Phi)^a} + \frac{(\cos \Phi - \sin \Phi)^{a-1}}{(\cos \Phi + \sin \Phi)^a} - 2 = O(\Phi^2).$$

Furthermore for $\Phi \in [0, \pi/4]$,

$$\begin{aligned} 0 &\leq \int_{[\epsilon/(4t)^{1/2}, \infty)} d\rho e^{-2\rho^2(\sin \Phi)^2} \leq (4t)^{1/2} \epsilon^{-1} \int_{[\epsilon/(4t)^{1/2}, \infty)} d\rho \rho e^{-2\rho^2(\sin \Phi)^2} \\ &\leq (4t)^{1/2} \epsilon^{-1} \int_{[0, \infty)} d\rho \rho e^{-2\rho^2(\sin \Phi)^2} = 2^{-1} t^{1/2} \epsilon^{-1} (\sin \Phi)^{-2}. \end{aligned}$$

Hence

$$|B_4(t)| \leq (2\pi)^{-1/2} t^{1/2} \epsilon^{-1} \int_{[0, \pi/4]} d\Phi (\sin \Phi)^{-2} \times \left| \frac{(\cos \Phi + \sin \Phi)^{a-1}}{(\cos \Phi - \sin \Phi)^a} + \frac{(\cos \Phi - \sin \Phi)^{a-1}}{(\cos \Phi + \sin \Phi)^a} - 2 \right|.$$

We see that the integral with respect to Φ converges both at $\Phi = 0$ and at $\Phi = \pi/4$. We conclude that $B_4 = O(t^{1/2})$. This gives the formula of Theorem 1.2 when $a + b = 1$.

3. THE 1-DIMENSIONAL SETTING: $\phi(x) = x^{-a}$ AND $\rho(x) = x^{-b}$

Let $\Re(a) < 1$ and $\Re(b) < 1$. Let $h_{a,b}^{[0,1]}(t)$ be the heat content of $[0, 1]$ in \mathbb{R} where $\phi(x) = x^{-a}$ and $\rho(x) = x^{-b}$. By Equation (1.c):

$$h_{a,b}^{[0,1]}(t) := \frac{1}{\sqrt{4\pi t}} \int_0^1 \int_0^1 e^{-(x-\tilde{x})^2/(4t)} x^{-a} \tilde{x}^{-b} dx d\tilde{x}. \quad (3.a)$$

Let $0 < t < 1$, let N be a non-negative integer with $N \geq -\Re(a+b) + 3$, and let $0 < \delta < \frac{1}{2}$. Let $\kappa(\cdot)$ be a bound which depends on the parameters indicated and which may increase a finite number of times in any given proof. Let \mathcal{O} be as in Equation (1.b) and let $c(a, b)$ be as in Theorem 1.1. Section 3 is devoted to the proof of the following special case of Theorem 1.1 that will be an essential step in our analysis of the general case.

Theorem 3.1. *Adopt the conventions established above. Let $(a, b) \in \mathcal{O}$, let $|(a, b)| \leq R$, and let $\text{dist}((a, b), \mathcal{O}^c) \geq \delta$. Let*

$$\begin{aligned} \theta_n(a) &:= \begin{cases} 1 & \text{if } n = 0 \\ a(a+1) \dots (a+n-1) & \text{if } n \geq 1 \end{cases}, \\ c_n(a, b) &:= \frac{(\theta_n(a) + \theta_n(b)) 2^{n-1} \Gamma(\frac{1+n}{2})}{\sqrt{\pi}(1-a-b-n)n!} \\ R_N(a, b) &:= \left| h_{a,b}^{[0,1]}(t) - \left\{ c(a, b) t^{(1-a-b)/2} + \sum_{n=0}^N c_n(a, b) t^{n/2} \right\} \right|. \end{aligned}$$

(1) *If $\Re(a+b) > -7$, then we have a uniform estimate of the form*

$$R_N(a, b) \leq \kappa(N, R)(1 + \delta^{-4}) \left\{ t^{(N+1)/2} + e^{-1/(36t)} \right\}.$$

(2) *If $\Re(a+b) \leq -7$, then for each (a, b) we have an estimate of the form*

$$R_N(a, b) \leq \mathcal{E}_N(a, b) t^{(N+1)/2}.$$

We shall need the uniform estimate of Assertion (1) for $0 < \Re(a+b) < 2$ in examining the log terms when $a+b = 1$, $a < 1$, and $a \in \mathbb{R}$ subsequently in Section 3.8; as this degree of precision is not needed for $\Re(a+b) \leq 0$, we have contented ourselves for the sake of brevity with a weaker estimate in Assertion (2).

We note that the terms in $t^{n/2}$ when n is odd can be regarded as arising from a boundary contribution at $x = 1$; the terms when n is even can be regarded as arising both from a boundary contribution at $x = 1$ and from regularized interior integrals. The term in $t^{(1-a-b)/2}$ can be regarded as arising from a boundary contribution at $x = 0$ and this is the only term which arises in this way.

Although a-priori $c(a, b)$ and $c_n(a, b)$ could have poles when $a + b = 0, -2, \dots$, we will show in Lemma 3.6 that they are regular on \mathcal{O} . We have, for example,

$$\begin{aligned} c_0(a, b) &= \frac{(1+1)2^{-1}\Gamma(\frac{1}{2})}{\sqrt{\pi}(1-a-b)} = \frac{1}{1-a-b}, & c_1(a, b) &= \frac{(a+b)2^0\Gamma(1)}{\sqrt{\pi}(-a-b)} = -\frac{1}{\sqrt{\pi}}, \\ c_2(a, b) &= -\frac{a^2 + a + b^2 + b}{2(1+a+b)}. \end{aligned} \quad (3.b)$$

Theorem 1.2 shows there are log terms when $a + b = 1$ and gives a very precise statement of what the terms are in the asymptotic expansion. In the following result, we give a complete asymptotic expansion with log terms when $a + b = 1 - 2k$ and when $(a, b) \in \mathbb{R}^2$. We shall identify the coefficient of the $\log(t)t^k$ term. We do not, however, identify the remaining coefficients explicitly although our methods would permit this. We must restrict to $(a, b) \in \mathbb{R}^2$ as we shall use a monotonicity argument; we do not know how to handle the case when $a + b = 1 - 2k$ and $\Im(a) \neq 0$.

Theorem 3.2. *Let $(a, b) \in \mathbb{R}^2$ satisfy $a < 1$, $b < 1$, and $a + b = 1 - 2k$ where k is a non-negative integer. There exist functions $\tilde{c}_n(a, b)$ which are real analytic in (a, b) so that if $N \geq 2k + 1$, then:*

$$h_{a,b}^{[0,1]}(t) = \sum_{n=0}^N \tilde{c}_n(a, b)t^{n/2} - \frac{a(a+1) \dots (a+2k-1)}{2 \cdot k!} \log(t)t^k + O(t^{(N+1)/2}).$$

Here is a brief outline to Section 3. In Section 3.1, we use Theorem 1.3 to show that the heat content functions of $[0, 1]$ in S^1 and of $[0, 1]$ in \mathbb{R} are the same up to an exponentially small error in t under very mild assumptions on ϕ and ρ (see Lemma 3.3). We then give two recursion relations that relate heat content functions for different values of $\{a, b\}$. These will be used subsequently in various induction arguments. Lemma 3.4 deals quite generally with heat content functions on the interval where the ambient manifold is S^1 and where ϕ and ρ satisfy certain vanishing and smoothness conditions; Lemma 3.5 deals with the heat content on the interval when $\phi(x) = x^{-a}$ and $\rho(x) = x^{-b}$ where the ambient manifold is \mathbb{R} . In Section 3.2 (see Lemma 3.6), we show that the function $c(a, b)$ of Theorem 1.1 and the functions $c_n(a, b)$ of Theorem 3.1 are holomorphic on \mathcal{O} despite the apparent poles when $a + b = -2k$ is an even non-positive integer. In Section 3.3 (see Lemma 3.7), we establish Theorem 3.1 in the special case that $b = 0, -1, -2, \dots$. Since ρ is smooth in this case, the asymptotic series follows from Theorem 1.4. However, we must reprove this result since we need a very precise control on the remainder terms. On the half line $[0, \infty)$, one is tempted to define:

$$\begin{aligned} h_{a,b}^{[0,\infty)}(t) &:= \frac{1}{\sqrt{4\pi t}} \int_0^\infty \int_0^\infty x^{-a} y^{-b} e^{-(x-y)^2/(4t)} dy dx \\ &= \frac{1}{\sqrt{4\pi t}} \int_0^\infty \int_0^x \{x^{-a} y^{-b} + x^{-b} y^{-a}\} e^{-(x-y)^2/(4t)} dy dx. \end{aligned} \quad (3.c)$$

Although the integral of Equation (3.a) converges for $\Re(a) < 1$ and $\Re(b) < 1$, the integral of Equation (3.c) only converges if we assume additionally that $\Re(a + b) > 1$. Thus we must regularize this integral for other points (a, b) of \mathcal{O} and define a regularized heat content function. In Section 3.4 (see Lemma 3.8), we prove a technical result that we will use to regularize the integral of Equation (3.c). This result is Mathematica [19] assisted and is the reason we assumed $-7 < \Re(a + b)$ in Theorem 3.1. In Section 3.5 (see Lemma 3.9), we study a regularized heat trace on the half line and in Section 3.6 (see Lemma 3.10), we use the regularized heat trace on the half line of Lemma 3.9 to study a corresponding regularized heat trace on the interval. The regularized heat trace of Lemma 3.10 together with the expansion of Lemma 3.7 is then used in Section 3.7 to complete the proof of Theorem 3.1; the recursion relations of Section 3.3 play a central role in this analysis. In Section 3.8, we begin the proof of Theorem 3.2 by examining the situation when $a + b = 1$, $a < 1$, and $a \in \mathbb{R}$. We complete the proof of Theorem 3.2 in Section 3.9 by using Lemma 3.4 once again.

3.1. Recursion relations. The following lemma follows immediately from Theorem 1.3.

Lemma 3.3. *Let $D_{\mathbb{R}} = -\partial_x^2$ on \mathbb{R} and let $D_{S^1} = -\partial_{x^2}$ on S^1 . Let $M = S^1$ or $M = \mathbb{R}$, and let $\tilde{M} = [-1, 2]$. If $\Re(a) \leq 0$ and $\Re(b) < 1$, then*

$$|\beta_{\Omega}(\phi, \rho, D_{S^1})(t) - \beta_{\Omega}^{\tilde{M}}(\phi, \rho, D_{\mathbb{R}})(t)| \leq 2^{(2+m)/2} \frac{1}{1 - \Re(b)} e^{-1/(8t)}.$$

We first suppose the ambient manifold is the circle S^1 . A *spectral resolution* for D_{S^1} and the corresponding *Fourier coefficients* for $\phi \in L^1(S^1)$ are given by

$$\left\{ \frac{e^{\sqrt{-1}n}}{\sqrt{2\pi}}, n^2 \right\}_{n=-\infty}^{\infty} \quad \text{and} \quad \gamma_n(\phi) := \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} \phi(x) e^{-\sqrt{-1}nx} dx.$$

If ϕ and ρ are in L^1 , then the Fourier coefficients are uniformly bounded and we have for $t > 0$ that:

$$e^{-tD_{S^1}} \phi = \sum_{n=-\infty}^{\infty} e^{-tn^2} \gamma_n(\phi) e^{\sqrt{-1}nx} \quad \text{and} \quad \beta(\phi, \rho, D_{S^1})(t) = \sum_{n=-\infty}^{\infty} e^{-tn^2} \gamma_n(\phi) \gamma_{-n}(\rho).$$

Let $C_0[0, 1]$ be the space of continuous functions on the interval which vanish at $x = 0$ and at $x = 1$. Let $C_0^1[0, 1]$ be the space of continuously differentiable functions ϕ on the interval so ϕ and ϕ' both vanish at $x = 0$ and at $x = 1$.

Lemma 3.4.

- (1) *If $\phi \in C^\infty(0, 1) \cap C_0[0, 1]$, and if $\phi' \in L^1[0, 1]$, then $\gamma_n(\phi') = \sqrt{-1}n\gamma_n(\phi)$.*
- (2) *If $\phi \in C^\infty(0, 1) \cap C_0^1[0, 1]$, if $\phi'' \in L^1[0, 1]$, and if $\rho \in L^1[0, 1]$, then:*
 - (a) $\beta_{[0,1]}(\phi'', \rho, D_{S^1})(t) = \partial_t \beta_{[0,1]}(\phi, \rho, D_{S^1})(t)$.
 - (b) $\beta_{[0,1]}(\phi, \rho, D_{S^1})(t)$ is continuous.
 - (c) $|\beta_{[0,1]}(\phi, \rho, D_{S^1})(t)| \leq \frac{2}{3}\pi^2 \|\phi''\|_{L^1(S^1)} \|\rho\|_{L^1(S^1)}$ for $t \in [0, \infty)$.
- (3) *If $\phi, \rho \in C^\infty(0, 1) \cap C_0[0, 1]$, if $\phi' \in L^1[0, 1]$, and if $\rho' \in L^1[0, 1]$, then:*
 - (a) $\beta_{[0,1]}(\phi', \rho', D_{S^1})(t) = -\partial_t \beta_{[0,1]}(\phi, \rho, D_{S^1})(t)$.
 - (b) $\beta_{[0,1]}(\phi, \rho, D_{S^1})(t)$ is continuous.
 - (c) $|\beta_{[0,1]}(\phi, \rho, D_{S^1})(t)| \leq \frac{2}{3}\pi^2 \|\phi'\|_{L^1(S^1)} \|\rho'\|_{L^1(S^1)}$ for $t \in [0, \infty)$.

Proof. Suppose $\phi \in C^\infty(0, 1) \cap C_0[0, 1]$ and that $\phi' \in L^1[0, 1]$. We establish Assertion (1) by computing:

$$\begin{aligned} & \lim_{\epsilon \downarrow 0} \int_{\epsilon}^{1-\epsilon} \partial_x \left\{ \phi e^{-\sqrt{-1}nx} dx \right\} dx = \lim_{\epsilon \downarrow 0} \left\{ \phi(x) e^{-\sqrt{-1}nx} \right\} \Big|_{x=\epsilon}^{1-\epsilon} = \left\{ \phi(x) e^{-\sqrt{-1}nx} \right\} \Big|_{x=0}^1 = 0 \\ & = \lim_{\epsilon \downarrow 0} \left\{ \int_{\epsilon}^{1-\epsilon} \phi'(x) e^{-\sqrt{-1}nx} - \sqrt{-1}n \phi(x) e^{-\sqrt{-1}nx} \right\} dx = \gamma_n(\phi') - n\sqrt{-1}\gamma_n(\phi). \end{aligned}$$

Let $\phi \in C^\infty(0, 1) \cap C_0^1[0, 1]$, let $\phi' \in L^1[0, 1]$, and let $\rho \in L^1[0, 1]$. We can use Assertion (1) to see that $\gamma_n(\phi'') = -n^2\gamma_n(\phi)$ and hence

$$\begin{aligned} \beta_{[0,1]}(\phi'', \rho, D_{S^1})(t) &= - \sum_n n^2 e^{-tn^2} \gamma_n(\phi) \gamma_n(\rho) = \partial_t \sum_n e^{-tn^2} \gamma_n(\phi) \gamma_{-n}(\rho) \\ &= \partial_t \beta_{[0,1]}(\phi, \rho, D_{S^1})(t). \end{aligned}$$

If $n \neq 0$, then $|\gamma_n(\phi)| \leq n^{-2} \|\phi''\|_{L^1(S^1)}$ and $|\gamma_n(\rho)| \leq \|\rho\|_{L^1(S^1)}$. This shows that the series defining $\beta_{[0,1]}(\phi, \rho, D_{S^1})(t)$ converges uniformly on $[0, \infty)$ and hence is continuous on $[0, \infty)$. We estimate:

$$|\beta_{[0,1]}(\phi, \rho, D_{S^1})(t)| \leq \|\phi''\|_{L^1(S^1)} \|\rho\|_{L^1(S^1)} \sum_{n \neq 0} \frac{1}{n^2} = \|\phi''\|_{L^1(S^1)} \|\rho\|_{L^1(S^1)} \frac{2}{3} \pi^2.$$

Similarly, if $\phi \in C^\infty(0, 1) \cap C_0[0, 1]$, if $\rho \in C^\infty(0, 1) \cap C_0[0, 1]$, if $\phi' \in L^1[0, 1]$, and if $\rho' \in L^1[0, 1]$, then:

$$\begin{aligned} \gamma_n(\phi') &= \sqrt{-1}n\gamma_n(\phi), \quad \gamma_{-n}(\rho') = -\sqrt{-1}n\gamma_{-n}(\rho), \\ \beta_{[0,1]}(\phi', \rho', D_{S^1})(t) &= + \sum_n n^2 e^{-tn^2} \gamma_n(\phi) \gamma_n(\rho) = -\partial_t \sum_n e^{-tn^2} \gamma_n(\phi) \gamma_n(\rho) \\ &= -\partial_t \beta_{[0,1]}(\phi, \rho, D_{S^1})(t). \end{aligned}$$

We have $\gamma_0(\phi'') = 0$. If $n \neq 0$, we may estimate $|\gamma_n(\rho)| \leq \frac{1}{n}|\phi'|_{L^1}$ and $|\gamma_n(\rho)| \leq \frac{1}{n}|\rho'|_{L^1}$. Thus the series defining $\beta_{[0,1]}(\phi, \rho, D_{S^1})(t)$ converges uniformly on $[0, \infty)$ and hence is continuous in $[0, \infty)$ and we have a similar estimate. \square

Next we assume that the ambient manifold is \mathbb{R} , that $\phi(x) = x^{-a}$, and that $\rho(x) = x^{-b}$. Let $\theta_n(b)$ be as in Theorem 3.1. Let $|(a, b)| = \{|a|^2 + |b|^2\}^{1/2}$.

Lemma 3.5. *Let $\Re(a) < 1 - \delta$, let $\Re(b) < 1 - \delta$, and let $|(a, b)| < R$. Then:*

$$\begin{aligned} &\left| h_{a,b}^{[0,1]}(t) - \frac{1}{2t(1-a)} \left\{ h_{a-2,b}^{[0,1]}(t) - h_{a-1,b-1}^{[0,1]}(t) - \frac{1}{\sqrt{\pi}} \sum_{n=0}^N t^{(n+1)/2} 2^n \frac{\theta_n(b)}{n!} \Gamma\left(\frac{n+2}{2}\right) \right\} \right| \\ &\leq \mathcal{E}_N(\delta, R, t). \end{aligned}$$

Proof. We integrate by parts in x with $u = \tilde{x}^{-b} e^{-(x-\tilde{x})^2/(4t)}$, $v = x^{1-a}/(1-a)$, and $dv = x^{-a} dx$. Since $(uv)(0) = 0$, we have that:

$$\begin{aligned} h_{a,b}^{[0,1]}(t) &= \frac{1}{\sqrt{4\pi t}} \int_0^1 \int_0^1 x^{-a} \tilde{x}^{-b} e^{-(x-\tilde{x})^2/(4t)} dx d\tilde{x} \\ &= \frac{1}{2t\sqrt{4\pi t}(1-a)} \left\{ \int_0^1 \int_0^1 x^{1-a} \tilde{x}^{-b} (x-\tilde{x}) e^{-(x-\tilde{x})^2/(4t)} dx d\tilde{x} - \int_0^1 \tilde{x}^{-b} (1-\tilde{x}) e^{-(1-\tilde{x})^2/(4t)} d\tilde{x} \right\} \\ &= \frac{1}{2t(1-a)} \left\{ h_{a-2,b}^{[0,1]}(t) - h_{a-1,b-1}^{[0,1]}(t) + \mathcal{Z}_{a,b}(t) \right\}. \end{aligned}$$

We replace \tilde{x} by $1 - \tilde{x}$ and set $u = \tilde{x}/\sqrt{4t}$ to see:

$$\mathcal{Z}_{a,b}(t) = -\frac{1}{\sqrt{4\pi t}} \int_0^1 (1-\tilde{x})^{-b} \tilde{x} e^{-\tilde{x}^2/(4t)} d\tilde{x} = -\frac{1}{\sqrt{\pi}} (4t)^{1/2} \int_0^{1/\sqrt{4t}} (1-\sqrt{4t}u)^{-b} u e^{-u^2} du.$$

Let θ_n be as in Theorem 3.1. We expand the function $(1-\eta)^{-b}$ in a Taylor series about $\eta = 0$:

$$(1-\eta)^{-b} = \sum_{n=0}^N \frac{\theta_n(b)}{n!} \eta^n + R_{N,b}(\eta) \quad (3.d)$$

to see

$$\mathcal{Z}_{a,b}(t) = -\frac{1}{\sqrt{\pi}} \sum_{n=0}^N (4t)^{(n+1)/2} \int_0^{1/\sqrt{4t}} \frac{\theta_n(b)}{n!} u^{n+1} e^{-u^2} du + O(t^N).$$

We may now replace the upper integral by ∞ modulo an exponentially small error and evaluate the coefficients to complete the proof by expanding

$$\mathcal{Z}_{a,b}(t) = -\frac{1}{\sqrt{\pi}} \sum_{n=0}^N t^{(n+1)/2} 2^n \frac{\theta_n(b)}{n!} \Gamma\left(\frac{n+2}{2}\right). \quad \square$$

3.2. Regularity of $c(a, b)$ and $c_n(a, b)$ when $a + b = 0, -2, -4, \dots$

Lemma 3.6. *The functions $c(a, b)$ and $c_n(a, b)$ of Equation (3.b) are holomorphic on \mathcal{O} .*

Proof. Only points (a, b) of \mathcal{O} where $a + b = -2k$ is a non-positive even integer are at issue. Since $\Re(a) + \Re(b) < 2$, $\Gamma(\frac{2-a-b}{2})$ is always regular. Since $\Re(a) < 1$, $\Re(b) < 1$, $\Gamma(1-a)/\Gamma(b)$ and $\Gamma(1-b)/\Gamma(a)$ are regular as well. Since $\Gamma(a+b-1)$ has a simple pole when $a+b = 1, 0, -1, -2, \dots$, we must show that

$$\frac{\Gamma(1-a)}{\Gamma(b)} + \frac{\Gamma(1-b)}{\Gamma(a)}$$

vanishes when $a+b = -2k$ is a non-positive even integer. The *Euler reflection formula* (see, for example, Formula 8.334.3 in [15]) implies:

$$\Gamma(\epsilon - 2k)\Gamma(2k + 1 - \epsilon) = -\Gamma(1 + \epsilon)\Gamma(-\epsilon).$$

Assume $a+b = -2k$ is a non-positive even integer. We set $\epsilon = -b$ to see:

$$\begin{aligned} \Gamma(a)\Gamma(1-a) + \Gamma(b)\Gamma(1-b) &= \Gamma(-2k-b)\Gamma(1+2k+b) + \Gamma(b)\Gamma(1-b) \\ &= \Gamma(-2k+\epsilon)\Gamma(1+2k-\epsilon) + \Gamma(-\epsilon)\Gamma(1+\epsilon) = \{1-1\}\Gamma(1+\epsilon)\Gamma(-\epsilon) = 0. \end{aligned}$$

The denominator $(1-a-b-n)^{-1}$ defining c_n has a simple pole when $a+b = -n+1$. Thus to show $c_n(a, b)$ is regular on \mathcal{O} , we must show $\theta_n(a) + \theta_n(b) = 0$ when $a+b = 1-n$ where n is an odd positive integer. Set $b = 1-n-a$. We have

$$\begin{aligned} \theta_n(a) + \theta_n(1-n-a) &= \{a(a+1)\dots(a+(n-1))\} + \{(1-n-a)\dots(-a)\} \\ &= \{1+(-1)^n\}a(a+1)\dots(a+(n-1)). \end{aligned}$$

This vanishes if n is odd. □

3.3. Heat content asymptotics if $m = 1$, if $\phi(x) = x^{-a}$, and if $\rho(x) = x^{-b}$ is smooth. The following special case will play a central role in our regularization procedure. As we need a very precise control on the remainder estimate, we can not simply appeal to Theorem 1.4.

Lemma 3.7. *Let $R > 0$, let $\delta > 0$, let b be a non-positive integer, and let N be a non-negative integer with $N \geq \Re(a+b) + 3$. There exist holomorphic functions $\{C_b(a), C_{b,n}(a)\}$ defined for $\Re(a) < 1$ and $n = 0, 1, \dots$ so that if $\Re(a) < 1 - \delta$ and if $|a| \leq R$, then:*

$$\begin{aligned} &\left| h_{a,b}^{[0,1]}(t) - \left\{ C_b(a)t^{(1-a-b)/2} + \sum_{n=0}^N C_{b,n}(a)t^{n/2} \right\} \right| \\ &\leq \kappa(b, N, R)(1 + \delta^{-1}) \left\{ t^{(N+1)/2} + t^{(1-b-\Re(a))/2} e^{-1/(36t)} \right\}. \end{aligned}$$

Proof. Let $\Xi \in C^\infty(\mathbb{R})$ be a smooth monotonically decreasing cut-off function satisfying

$$\Xi(x) = 1 \text{ for } x \leq \frac{1}{3} \quad \text{and} \quad \Xi(x) = 0 \text{ for } x \geq \frac{2}{3}. \quad (3.e)$$

The choice of Ξ will, of course, play no role in the final formulas. Express

$$\begin{aligned} h_{a,b}^{[0,1]}(t) &= \frac{1}{\sqrt{4\pi t}} \int_0^1 \int_0^1 \Xi(x) x^{-b} \tilde{x}^{-a} e^{-(x-\tilde{x})^2/(4t)} dx d\tilde{x} \\ &\quad + \frac{1}{\sqrt{4\pi t}} \int_0^1 \int_0^1 (1-\Xi(x)) x^{-b} \tilde{x}^{-a} e^{-(x-\tilde{x})^2/(4t)} dx d\tilde{x}. \end{aligned}$$

We replace x by $1-x$ and \tilde{x} by $1-\tilde{x}$ to see:

$$h_{a,b}^{[0,1]}(t) = \frac{1}{\sqrt{4\pi t}} \int_0^1 \int_0^1 \{ \Xi(x) x^{-b} \tilde{x}^{-a} + (1-\Xi(1-x))(1-x)^{-b}(1-\tilde{x})^{-a} \} e^{-(x-\tilde{x})^2/(4t)} dx d\tilde{x}.$$

Since $\Xi(x)$ and $(1-\Xi(1-x))$ vanish for $x \geq 1$, we may let the dx integral range from 1 to ∞ :

$$h_{a,b}^{[0,1]}(t) = \frac{1}{\sqrt{4\pi t}} \int_0^1 \int_0^\infty \{ \Xi(x) x^{-b} \tilde{x}^{-a} + (1-\Xi(1-x))(1-x)^{-b}(1-\tilde{x})^{-a} \} e^{-(x-\tilde{x})^2/(4t)} dx d\tilde{x}.$$

Recall that b is a non-positive integer, and decompose:

$$h_{a,b}^{[0,1]}(t) = \mathcal{A}_{a,b}(t) + \mathcal{B}_{a,b}(t) + \mathcal{C}_{a,b}(t) \text{ where}$$

$$\begin{aligned}
\mathcal{A}_{a,b}(t) &:= \frac{1}{\sqrt{4\pi t}} \int_0^1 \int_{-\infty}^{\infty} \Xi(x) x^{-b} \tilde{x}^{-a} e^{-(x-\tilde{x})^2/(4t)} dx d\tilde{x} \\
&\quad + \frac{1}{\sqrt{4\pi t}} \int_0^1 \int_{-\infty}^{\infty} (1 - \Xi(1-x))(1-x)^{-b} (1-\tilde{x})^{-a} e^{-(x-\tilde{x})^2/(4t)} dx d\tilde{x}, \\
\mathcal{B}_{a,b}(t) &:= -\frac{1}{\sqrt{4\pi t}} \int_0^1 \int_{-\infty}^0 \Xi(x) x^{-b} \tilde{x}^{-a} e^{-(x-\tilde{x})^2/(4t)} dx d\tilde{x}, \\
\mathcal{C}_{a,b}(t) &:= -\frac{1}{\sqrt{4\pi t}} \int_0^1 \int_{-\infty}^0 (1 - \Xi(1-x))(1-x)^{-b} (1-\tilde{x})^{-a} e^{-(x-\tilde{x})^2/(4t)} dx d\tilde{x}.
\end{aligned}$$

The two integrals defining $\mathcal{A}_{a,b}(t)$ for $x \leq 0$ and the integrals defining $\mathcal{B}_{a,b}(t)$ and $\mathcal{C}_{a,b}(t)$ converge absolutely at $x = -\infty$ owing to the exponential damping $e^{-(x-\tilde{x})^2/(4t)} \leq e^{-(x^2+\tilde{x}^2)/(4t)}$ for $x \leq 0$. We again replace x by $1-x$ and \tilde{x} by $1-\tilde{x}$ in the second integral defining $\mathcal{A}_{a,b}(t)$ and use the fact that $\Xi(x) + (1 - \Xi(x)) = 1$ to express:

$$\mathcal{A}_{a,b}(t) = \frac{1}{\sqrt{4\pi t}} \int_0^1 \int_{-\infty}^{\infty} x^{-b} \tilde{x}^{-a} e^{-(x-\tilde{x})^2/(4t)} dx d\tilde{x}. \quad (3.f)$$

We replace x by $-x$. Since $\Xi(x) = 1$ for $x \leq 0$ and $\Xi(1-x) = 0$ for $x \leq 0$, we have:

$$\mathcal{B}_{a,b}(t) = (-1)^{b+1} \frac{1}{\sqrt{4\pi t}} \int_0^1 \int_0^{\infty} x^{-b} \tilde{x}^{-a} e^{-(x+\tilde{x})^2/(4t)} dx d\tilde{x}, \quad (3.g)$$

$$\mathcal{C}_{a,b}(t) = -\frac{1}{\sqrt{4\pi t}} \int_0^1 \int_0^{\infty} (1+x)^{-b} (1-\tilde{x})^{-a} e^{-(x+\tilde{x})^2/(4t)} dx d\tilde{x}. \quad (3.h)$$

We complete the proof by expanding each of these integrals separately. We change variables and set $u = (x - \tilde{x})/\sqrt{4t}$ in Equation (3.f) to express:

$$\mathcal{A}_{a,b}(t) = \frac{1}{\sqrt{\pi}} \int_0^1 \int_{-\infty}^{\infty} (\tilde{x} + \sqrt{4t}u)^{-b} \tilde{x}^{-a} e^{-u^2} du d\tilde{x}.$$

Since $-b$ is a non-negative integer, we may expand $(\tilde{x} + \sqrt{4t}u)^{-b}$ using the binomial theorem:

$$(\tilde{x} + \sqrt{4t}u)^{-b} = \sum_{n=0}^{-b} \frac{(-b)!}{(-b-n)!n!} \tilde{x}^{-b-n} (4t)^{n/2} u^n.$$

This shows that

$$\mathcal{A}_{a,b}(t) = \frac{1}{\sqrt{\pi}} \sum_{n=0}^{-b} (4t)^{n/2} \frac{(-b)!}{(-b-n)!n!} \int_0^1 \tilde{x}^{-a-b-n} d\tilde{x} \cdot \int_{-\infty}^{\infty} u^n e^{-u^2} du.$$

Since $-b-n \geq 0$ and $\Re(a) < 1$, the integrals converge absolutely to define holomorphic functions of a . This gives rise to an expansion of the form given in Lemma 3.7. The remainder term is identically equal to 0.

Next, we examine $\mathcal{B}_{a,b}(t)$. We interchange the order of integration and replace x by $\sqrt{4t}x$ and \tilde{x} by $\sqrt{4t}\tilde{x}$ in Equation (3.g) to express:

$$\mathcal{B}_{a,b}(t) := t^{(1-a-b)/2} (-1)^{b+1} \frac{2^{(1-a-b)}}{\sqrt{\pi}} \int_0^{1/\sqrt{4t}} \int_0^{\infty} x^{-b} \tilde{x}^{-a} e^{-(x+\tilde{x})^2} dx d\tilde{x}.$$

Let

$$C_b(a) := (-1)^{b+1} \frac{2^{(1-a-b)}}{\sqrt{\pi}} \int_0^{\infty} \int_0^{\infty} x^{-b} \tilde{x}^{-a} e^{-(x+\tilde{x})^2} dx d\tilde{x}.$$

We decompose $\mathcal{B}_{a,b}(t) = \mathcal{B}_{a,b}^1(t) - \mathcal{B}_{a,b}^2(t)$ where

$$\begin{aligned}\mathcal{B}_{a,b}^1(t) &:= t^{(1-a-b)/2} (-1)^{b+1} \frac{2^{(1-a-b)}}{\sqrt{\pi}} \int_0^\infty \int_0^\infty x^{-b} \tilde{x}^{-a} e^{-(x+\tilde{x})^2} dx d\tilde{x} \\ &= t^{(1-a-b)/2} C_b(a), \\ \mathcal{B}_{a,b}^2(t) &:= t^{(1-a-b)/2} (-1)^{b+1} \frac{2^{(1-a-b)}}{\sqrt{\pi}} \int_{1/\sqrt{4t}}^\infty \int_0^\infty x^{-b} \tilde{x}^{-a} e^{-(x+\tilde{x})^2} dx d\tilde{x}.\end{aligned}$$

If $x \geq \frac{1}{\sqrt{4t}}$, then $e^{-(x+\tilde{x})^2} \leq e^{-1/(8t)} e^{-(x^2+\tilde{x}^2)/2}$. This permits us to estimate:

$$|\mathcal{B}_{a,b}^2(t)| \leq t^{(1-\Re(a+b))/2} e^{-1/(8t)} \frac{2^{1-\Re(a+b)}}{\sqrt{\pi}} \int_0^\infty x^{-b} e^{-x^2/2} dx \cdot \int_0^\infty \tilde{x}^{-\Re(a)} e^{-\tilde{x}^2/2} d\tilde{x}.$$

The constant $|\pi^{-1/2} 2^{(1-a-b)}|$ can be uniformly bounded since $|a| \leq R$. The dx integral is convergent since $-b \geq 0$. Since $|a| \leq R$, the $d\tilde{x}$ integral is convergent on $[1, \infty)$ and can be estimated uniformly by some suitably chosen constant $\kappa(R)$. The $d\tilde{x}$ integral also is convergent on $[0, 1]$ since $\Re(a) \leq 1 - \delta$ implies $-\Re(a) \geq \delta - 1$ and consequently can be estimated uniformly by δ^{-1} . This gives rise to a remainder term of the form given.

Finally, we replace x and \tilde{x} by $\sqrt{4t}x$ and $\sqrt{4t}\tilde{x}$ in Equation (3.h) to express

$$\mathcal{C}_{a,b}(t) := -\frac{\sqrt{4t}}{\sqrt{\pi}} \int_0^{1/\sqrt{4t}} \int_0^\infty (1 + \sqrt{4t}x)^{-b} (1 - \sqrt{4t}\tilde{x})^{-a} e^{-(x+\tilde{x})^2} dx d\tilde{x}.$$

We use the binomial theorem to expand $(1 + \sqrt{4t}x)^{-b}$ in powers of $\sqrt{4t}x$. Thus, after suppressing the normalizing constants (which can be uniformly bounded), it suffices to consider an integral of the form

$$\mathcal{C}_{a,b,k}(t) = t^{(1+k)/2} \int_0^{1/\sqrt{4t}} \int_0^\infty x^k (1 - \sqrt{4t}\tilde{x})^{-a} e^{-(x+\tilde{x})^2} dx d\tilde{x} \text{ for } 0 \leq k \leq -b.$$

We decompose $\mathcal{C}_{a,b,k}(t) = \mathcal{C}_{a,b,k}^1(t) + \mathcal{C}_{a,b,k}^2(t)$ where

$$\begin{aligned}\mathcal{C}_{a,b,k}^1(t) &= t^{(1+k)/2} \int_0^{1/(3\sqrt{4t})} \int_0^\infty x^k (1 - \sqrt{4t}\tilde{x})^{-a} e^{-(x+\tilde{x})^2} dx d\tilde{x}, \\ \mathcal{C}_{a,b,k}^2(t) &= t^{(1+k)/2} \int_{1/(3\sqrt{4t})}^{1/\sqrt{4t}} \int_0^\infty x^k (1 - \sqrt{4t}\tilde{x})^{-a} e^{-(x+\tilde{x})^2} dx d\tilde{x}.\end{aligned}$$

As before, we see that the integral defining $\mathcal{C}_{a,b,k}^2$ is exponentially damped and satisfies an estimate

$$|\mathcal{C}_{a,b,k}^2| \leq \kappa(b, R)(1 + \delta^{-1})e^{-1/(36t)}.$$

Expand $(1 - w)^{-a}$ in a Taylor series for $|w| \leq \frac{2}{3}$ to see:

$$\left| (1 - w)^{-a} - \sum_{n=0}^N \frac{\theta_n(a)}{n!} w^n \right| \leq \kappa(N, R)|w|^{N+1}$$

for suitably defined holomorphic functions $\theta_n(a)$. Replacing w by $\sqrt{4t}\tilde{x}$ and noting $|\sqrt{4t}\tilde{x}| \leq \frac{1}{3}$ for $\tilde{x} \in [0, 1/(3\sqrt{4t})]$, then leads to an expansion of the desired form. \square

3.4. Regularizing the integrals on \mathbb{R} . Let $\mathcal{U} := \{(a, b) \in \mathbb{C}^2 : \Re(a) < 1 \text{ and } \Re(b) < 1\}$. Let

$$\begin{aligned}\mathcal{O}_{-1} &:= \{(a, b) \in \mathcal{U} : 1 < \Re(a + b) < 2\}, \\ \mathcal{O}_0 &:= \{(a, b) \in \mathcal{U} : -1 < \Re(a + b) < 1\}, \\ \mathcal{O}_1 &:= \{(a, b) \in \mathcal{U} : -3 < \Re(a + b) < 0 \text{ and } a + b \neq -1\}, \\ \mathcal{O}_2 &:= \{(a, b) \in \mathcal{U} : -5 < \Re(a + b) < -1 \text{ and } a + b \neq -1, -2, -3\} \\ \mathcal{O}_3 &:= \{(a, b) \in \mathcal{U} : -7 < \Re(a + b) < -2 \text{ and } a + b \neq -1, -2, -3, -4, -5\}.\end{aligned}$$

We have $0 \in \mathcal{O}_0$, $-2 \in \mathcal{O}_1$, $-4 \in \mathcal{O}_2$, $-6 \in \mathcal{O}_3$. Furthermore

$$\mathcal{O}_k \cap \mathcal{O}_{k+1} \neq \emptyset \text{ for } k \geq 0 \text{ and } \bigcup_{k=0}^3 \mathcal{O}_k = \{(a, b) \in \mathcal{O} : -7 < \Re(a + b) < 1\}.$$

Lemma 3.8. *Let $-1 \leq k \leq 3$, let $R > 0$, and let $0 < \delta < \frac{1}{2}$. There exist holomorphic functions $\sigma_{k,\ell}$ which are defined on \mathcal{O}_k so that if we define*

$$G_k(\eta; a, b) := \eta^{-a} + \eta^{-b} - \sum_{\ell=0}^k \sigma_{k,\ell}(a, b) \{\eta^\ell + \eta^{-a-b-\ell}\}$$

then if $(a, b) \in \mathcal{O}_k$ with $|(a, b)| \leq R$ and $\text{dist}((a, b), \mathcal{O}_k^c) \geq \delta$, then:

- (1) $|G_k(\eta; a, b)| \leq \kappa(k, R)(1 + \delta^{-k})(1 - |\eta|)^{2k+2}$ for $\eta \in [\frac{1}{2}, 1]$.
- (2) $|G_k(\eta; a, b)| \leq \kappa(k, R)(1 + \delta^{-k})\eta^{\delta-1}$ for $\eta \in [0, \frac{1}{2}]$.

Proof. Suppose first that $k = -1$ so the sum over ℓ does not appear. Clearly $|\eta^{-a} + \eta^{-b}|$ is bounded on $[\frac{1}{2}, 1]$ so Assertion (1) holds. Since $\Re(a) \leq 1 - \delta$ and $\Re(b) \leq 1 - \delta$ we have $|\eta^{-a} + \eta^{-b}| \leq 2\eta^{1-\delta}$ on $[0, \frac{1}{2}]$ and Assertion (2) holds. The Lemma now follows in this special case.

We introduce some additional notation to handle the other values of k . For i a non-negative integer, set:

$$\begin{aligned} A_i(\eta) &:= \eta^{-a-b-i} + \eta^i, & B_i(\eta) &:= \frac{2(A_i(\eta) - A_{i+1}(\eta))}{(-2i-1-a-b)(1-\eta)^2}, \\ C_i(\eta) &:= \frac{6(B_i(\eta) - B_{i+1}(\eta))}{(-2i-2-a-b)(1-\eta)^2}, & D_i(\eta) &:= \frac{10(C_i(\eta) - C_{i+1}(\eta))}{(-2i-3-a-b)(1-\eta)^2}, \\ E_i(\eta) &:= \frac{14(D_i(\eta) - D_{i+1}(\eta))}{(-2i-4-a-b)(1-\eta)^2}, & F_i(\eta) &:= \frac{18(E_i(\eta) - E_{i+1}(\eta))}{(-2i-5-a-b)(1-\eta)^2}, \\ G_i(\eta) &:= \frac{22(F_i(\eta) - F_{i+1}(\eta))}{(-2i-6-a-b)(1-\eta)^2}, & H_i(\eta) &:= \frac{26(G_i(\eta) - G_{i+1}(\eta))}{(-2i-7-a-b)(1-\eta)^2}, \\ X_0(\eta) &:= \frac{A_0(\eta) - \eta^{-a} - \eta^{-b}}{ab(\eta-1)^2}, & X_1(\eta) &:= \frac{12(X_0(\eta) - \frac{1}{2}B_0(\eta))}{(-1-a-b-ab)(\eta-1)^2}, \\ X_2(\eta) &:= \frac{30(X_1(\eta) - \frac{1}{2}C_0(\eta))}{(-4-2a-2b-ab)(\eta-1)^2}, & X_3(\eta) &:= \frac{56(X_2(\eta) - \frac{1}{2}D_0(\eta))}{(-9-3a-3b-ab)(\eta-1)^2}, \\ X_4(\eta) &:= \frac{90(X_3(\eta) - \frac{1}{2}E_0(\eta))}{(-16-4a-4b-ab)(\eta-1)^2}, & X_5(\eta) &:= \frac{132(X_4(\eta) - \frac{1}{2}F_0(\eta))}{(-25-5a-5b-ab)(\eta-1)^2}, \\ X_6(\eta) &:= \frac{182(X_5(\eta) - \frac{1}{2}G_0(\eta))}{(-36-6a-6b-ab)(\eta-1)^2}, & X_7(\eta) &:= \frac{240(X_6(\eta) - \frac{1}{2}H_0(\eta))}{(-49-7a-7b-ab)(\eta-1)^2}. \end{aligned}$$

We used Mathematica [19] to see that all these functions are regular at $\eta = 1$ and involve coefficients which are polynomial in $\{a, b\}$; the divisions defining these expansions are exact. Suppose first $k = 0$. Then we may expand:

$$\eta^{-a} - \eta^{-b} - A_0(\eta) = -ab(\eta-1)^2 X_0(\eta). \quad (3.i)$$

This gives an error of $\kappa_0(R)|1-\eta|^2$. There are no denominators in the regularization; the coefficients are regular for all $(a, b) \in \mathbb{C}^2$ and, in particular, are holomorphic on \mathcal{O}_0 . Next suppose $k = 1$. We have $\text{Span}\{A_0, B_0\} = \text{Span}\{A_0, A_1\}$. The denominator $\{-1-a-b\}$ appears in B_0 . We have explicitly avoided $a+b = -1$ so this is holomorphic on \mathcal{O}_1 and the denominator can be bounded by δ^{-1} . We combine the expansion in Equation (3.i) with the expansion

$$X_0(\eta) = \frac{1}{2}B_0(\eta) + \frac{-1-a-b-ab}{12}(\eta-1)^2 X_1(\eta) \quad (3.j)$$

to obtain an expansion of $\eta^{-a} + \eta^{-b}$ in $\{A_0, A_1\}$ with an error of $\kappa_1(R)(1 + \delta^{-1})|\eta - 1|^4$. Next, we suppose $k = 2$. We have

$$\text{Span}\{A_0, B_0, C_0\} = \text{Span}\{A_0, B_0, B_1\} = \text{Span}\{A_0, A_1, A_2\}.$$

The denominators which appear are $(-1 - a - b)^{-1}$ in B_0 , $(-3 - a - b)^{-1}$ in B_1 , and $(-2 - a - b)^{-1}$ in C_0 . We have excluded these values from \mathcal{O}_2 and the denominators can be bounded by δ^{-1} on \mathcal{O}_2 . We combine the expansion in Equation (3.j) with the expansion

$$X_1(\eta) = \frac{1}{2}C_0 + \frac{(-4 - 2a - 2b - ab)}{30}(\eta - 1)^2 X_2(\eta) \quad (3.k)$$

to obtain an expansion of $\eta^{-a} + \eta^{-b}$ in $\{A_0, A_1, A_2\}$ with an error of $\kappa_2(R)(1 + \delta^{-1})|\eta - 1|^6$. Finally, suppose $k = 3$. We use $\{A_0, B_0, C_0, D_0\}$ or equivalently $\{A_0, A_1, A_2, A_3\}$. We have the denominators $\{(-1 - a - b)^{-1}, (-3 - a - b)^{-1}, (-5 - a - b)^{-1}\}$ in $\{B_0, B_1, B_2\}$, $\{(-2 - a - b)^{-1}, (-4 - a - b)^{-1}\}$ in $\{C_0, C_1\}$, and $(-3 - a - b)^{-1}$ in D_1 . We have excluded these values from \mathcal{O}_3 and the coefficients can be bounded by δ^{-2} since the denominator $(-3 - a - b)^{-1}$ appears twice. We combine the expansion in Equation (3.k) with the expansion

$$X_2(\eta) = \frac{1}{2}D_0 + \frac{(-9 - 3a - 3b - ab)}{56}(\eta - 1)^2 X_3(\eta)$$

to derive an expansion of $\eta^{-a} + \eta^{-b}$ in $\{A_0, A_1, A_2, A_3\}$ with an error of $\kappa_3(R)(1 + \delta^{-2})|\eta - 1|^8$. We could continue to iterate this process to $k = 7$ where the power of δ will grow gradually. The bound at the origin in all cases will follow from the requirement that $\Re(a + b) \leq -k + 1 - \delta$ since the greatest growth in the expansion will occur when $\ell = k$ and we take η^{-a-b-k} ; the bound $\Re(a + b) \geq -2k - 1 + \delta$ will be used later. \square

3.5. A regularized heat content function on the half line. The integral of Equation (3.c) converges exponentially away from the line $x = \tilde{x}$ but must be regularized near $x = \tilde{x}$ to ensure convergence.

Lemma 3.9. *Let $-1 \leq k \leq 3$, let $(a, b) \in \mathcal{O}_k$, and $\sigma_{k,\ell}$ be as in Lemma 3.8. Define:*

$$F_{-1}(x, \tilde{x}; a, b) := x^{-a}\tilde{x}^{-b} + x^{-b}\tilde{x}^{-a},$$

$$F_k(x, \tilde{x}; a, b) := x^{-a}\tilde{x}^{-b} + x^{-b}\tilde{x}^{-a} - \sum_{\ell=0}^k \sigma_{k,\ell}(a, b) \{x^{-a-b-\ell}\tilde{x}^\ell + x^\ell\tilde{x}^{-a-b-\ell}\} \text{ for } k \geq 0,$$

$$h_{a,b,k}^{[0,\infty)}(t) := \frac{1}{\sqrt{4\pi t}} \int_0^\infty \int_0^x F_k(x, \tilde{x}; a, b) e^{-(x-\tilde{x})^2/(4t)} d\tilde{x} dx.$$

- (1) *The integral defining $h_{a,b,k}^{[0,\infty)}(t)$ converges absolutely.*
- (2) *There exists a holomorphic function $C_k(a, b)$ on \mathcal{O}_k so that $h_{a,b,k}^{[0,\infty)}(t) = C_k(a, b)t^{(1-a-b)/2}$.*
- (3) *$C_{-1}(a, b) = c(a, b)$ where $c(a, b)$ is as given in Theorem 1.1.*

Proof. We postpone for the moment the proof of Assertion (1) and proceed formally. We replace x by $\sqrt{4t}x$ and \tilde{x} by $\sqrt{4t}\tilde{x}$ and use the fact that $F_k(\lambda x, \lambda \tilde{x}) = \lambda^{-a-b}F_k(x, \tilde{x})$ to express

$$h_{a,b,k}^{[0,\infty)}(t) = \frac{1}{\sqrt{\pi}}(4t)^{(1-a-b)/2}C_k(a, b), \text{ where}$$

$$C_k(a, b) := \frac{2^{1-a-b}}{\sqrt{\pi}} \int_0^\infty \int_0^x F_k(x, \tilde{x}; a, b) e^{-(x-\tilde{x})^2} d\tilde{x} dx.$$

Set $\tilde{x} = x\eta$. Let G_k be as in Lemma 3.8. This yields:

$$\begin{aligned} C_k(a, b) &= \frac{2^{1-a-b}}{\sqrt{\pi}} \int_0^\infty \int_0^1 x^{1-a-b} G_k(\eta; a, b) e^{-x^2(1-\eta)^2} d\eta dx \\ &= \frac{2^{1-a-b}}{\sqrt{\pi}} \int_0^1 \int_0^\infty x^{1-a-b} G_k(\eta; a, b) e^{-x^2(1-\eta)^2} dx d\eta. \end{aligned}$$

Change variables $x = \xi^{1/2}(1 - \eta)^{-1}$ to see

$$\begin{aligned} C_k(a, b) &= \frac{1}{\sqrt{\pi}} 2^{-a-b} \int_0^1 \int_0^\infty \xi^{(-a-b)/2} e^{-\xi} G_k(\eta; a, b) (1 - \eta)^{a+b-2} d\xi d\eta \\ &= \frac{1}{\sqrt{\pi}} 2^{-a-b} \Gamma\left(\frac{2-a-b}{2}\right) \int_0^1 G_k(\eta; a, b) (1 - \eta)^{a+b-2} d\eta. \end{aligned} \quad (3.1)$$

Since $-2k - 1 < \Re(a + b)$, Lemma 3.8 (1) implies $|G_k(\eta; a, b)|$ is integrable on $[\frac{1}{2}, 1]$. Lemma 3.8 (2) implies $|G_k(\eta; a, b)|$ is integrable on $[0, \frac{1}{2}]$. This proves Assertion (1) and Assertion (2). Suppose $k = -1$ so there are no regularizing terms. If $\Re(a + b) > 1$, we use Mathematica [19] to complete the proof by computing:

$$C_{-1}(a, b) = \frac{1}{\sqrt{\pi}} 2^{-a-b} \Gamma\left(\frac{2-a-b}{2}\right) \int_0^1 \{\eta^{-a} + \eta^{-b}\} (1 - \eta)^{a+b-2} d\eta = c(a, b) \quad \square$$

3.6. A regularized heat content function on the interval. Adopt the notation established previously to define $F_k(x, y; a, b)$, $G_k(\eta; a, b)$, $c(a, b)$, and $C_k(a, b)$. Let

$$\mathcal{E}_N(\delta, R, t) = \kappa(N, R)(1 + \delta^{-4}) \left\{ t^{(N+1)/2} + e^{-1/(36t)} \right\}.$$

Lemma 3.10. *Let $-1 \leq k \leq 4$. Let N be a non-negative integer. If $(a, b) \in \mathcal{O}_k$, let*

$$h_{a,b,k}^{[0,1]}(t) := \frac{1}{\sqrt{4\pi t}} \int_0^1 \int_0^x F_k(x, \tilde{x}; a, b) e^{-(x-\tilde{x})^2/(4t)} d\tilde{x} dx.$$

Let $C_k(a, b)$ be as given in Equation (3.1). There exist functions $C_{k,n}^1(a, b)$, $C_n^2(a, b)$, and $C_{k,n}^3(a, b)$ which are holomorphic on \mathcal{O} and there exists a bound $\kappa(N, R)$ so that if $(a, b) \in \mathcal{O}_k$, if $|(a, b)| \leq R$, and if $\text{dist}((a, b), \mathcal{O}_k^c) \geq \delta$, then

$$\left| h_{a,b,k}^{[0,1]}(t) - \left\{ C_k(a, b) t^{(1-a-b)/2} + \sum_{n=0}^N C_{k,n}^1(a, b) t^{n/2} \right\} \right| \leq \mathcal{E}_N(\delta, R, t).$$

In the special case that $k = -1$, we have that $C_{-1}(a, b) = c(a, b)$ and $C_{-1,n}^1(a, b) = c_n(a, b)$.

Proof. We compare the situation on $[0, 1]$ with the situation on $[0, \infty)$. Set

$$\begin{aligned} \mathcal{H}_{a,b,k}(t) &= h_{a,b,k}^{[0,\infty)}(t) - h_{a,b,k}^{[0,1]}(t) \\ &= \frac{1}{\sqrt{4\pi t}} \int_0^\infty \int_0^x \{1 - \chi_{[0,1]}(x) \chi_{[0,1]}(\tilde{x})\} F_k(x, \tilde{x}; a, b) e^{-(x-\tilde{x})^2/(4t)} d\tilde{x} dx. \end{aligned}$$

If $\tilde{x} \leq x$, then $\chi_{[0,1]}(x) \chi_{[0,1]}(\tilde{x}) = \chi_{[0,1]}(x)$. Consequently

$$\begin{aligned} \mathcal{H}_{a,b,k}(t) &= \frac{1}{\sqrt{4\pi t}} \int_0^\infty \int_0^x \{1 - \chi_{[0,1]}(x)\} F_k(x, \tilde{x}; a, b) e^{-(x-\tilde{x})^2/(4t)} d\tilde{x} dx \\ &= \frac{1}{\sqrt{4\pi t}} \int_1^\infty \int_0^x F_k(x, \tilde{x}; a, b) e^{-(x-\tilde{x})^2/(4t)} d\tilde{x} dx. \end{aligned}$$

Decompose $\mathcal{H}_{a,b,k}(t) = \mathcal{H}_{a,b,k}^{[0, \frac{1}{2}]}(t) + \mathcal{H}_{a,b,k}^{[\frac{1}{2}, 1]}(t)$ where

$$\begin{aligned} \mathcal{H}_{a,b,k}^{[0, \frac{1}{2}]}(t) &:= \frac{1}{\sqrt{4\pi t}} \int_1^\infty \int_0^{\frac{1}{2}x} F_k(x, \tilde{x}; a, b) e^{-(x-\tilde{x})^2/(4t)} d\tilde{x} dx, \\ \mathcal{H}_{a,b,k}^{[\frac{1}{2}, 1]}(t) &:= \frac{1}{\sqrt{4\pi t}} \int_1^\infty \int_{\frac{1}{2}x}^x F_k(x, \tilde{x}; a, b) e^{-(x-\tilde{x})^2/(4t)} d\tilde{x} dx. \end{aligned}$$

We now show that the integral defining $\mathcal{H}_{a,b,k}^{[0,\frac{1}{2}]}(t)$ converges and decays exponentially as $t \downarrow 0$. Replace x by $\sqrt{4t}x$ and \tilde{x} by $\sqrt{4t}\tilde{x}$ and use the fact that $F_k(cx, c\tilde{x}; a, b) = c^{-a-b}F(x, \tilde{x}; a, b)$ to see:

$$\begin{aligned} \left| \mathcal{H}_{a,b,k}^{[0,\frac{1}{2}]}(t) \right| &\leq \frac{1}{\sqrt{4\pi t}} \int_1^\infty \int_0^{\frac{1}{2}x} |F_k(x, \tilde{x}; a, b)| e^{-(x-\tilde{x})^2/(4t)} d\tilde{x} dx \\ &\leq \frac{1}{\sqrt{\pi}} (4t)^{\Re(1-a-b)/2} \int_{\frac{1}{\sqrt{4t}}}^\infty \int_0^{\frac{1}{2}x} |F_k(x, \tilde{x}; a, b)| e^{-(x-\tilde{x})^2} d\tilde{x} dx. \end{aligned} \quad (3.m)$$

Recall that:

$$\begin{aligned} G_k(\eta; a, b) &= \eta^{-a} + \eta^{-b} - \sum_{\ell=0}^k \sigma_{k,\ell}(a, b) \{ \eta^\ell + \eta^{-a-b-\ell} \}, \\ F_k(x, \tilde{x}; a, b) &= x^{-a} \tilde{x}^{-b} + x^{-b} \tilde{x}^{-a} - \sum_{\ell=0}^k \sigma_{k,\ell}(a, b) \{ x^{-a-b-\ell} \tilde{x}^\ell + x^\ell \tilde{x}^{-a-b-\ell} \}. \end{aligned}$$

We set $\tilde{x} = \eta x$ and change variables in Equation (3.m) to see:

$$\left| \mathcal{H}_{a,b,k}^{[0,\frac{1}{2}]}(t) \right| \leq \frac{1}{\sqrt{\pi}} (4t)^{\Re(1-a-b)/2} \int_{\frac{1}{\sqrt{4t}}}^\infty \int_0^{\frac{1}{2}} x^{\Re(1-a-b)} |G_k(\eta; a, b)| e^{-x^2(1-\eta)^2} d\eta dx.$$

By Lemma 3.8 (2), $|G_k(\eta; a, b)| \leq \kappa(k, R)(1 + \delta^{-4})\eta^{\delta-1}$ on $[0, \frac{1}{2}]$. We may estimate

$$e^{-x^2(1-\eta)^2} \leq e^{-x^2(1-\frac{1}{2})^2} = e^{-x^2/8} e^{-x^2/8} \leq e^{-1/(32t)} e^{-x^2/8}$$

for $0 \leq \eta \leq \frac{1}{2}$ and $\frac{1}{\sqrt{4t}} \leq x < \infty$. We show that $\mathcal{H}_{a,b,k}^{[0,\frac{1}{2}]}(t)$ is exponentially damped as $t \downarrow 0$ by estimating:

$$\begin{aligned} \left| \mathcal{H}_{a,b,k}^{[0,\frac{1}{2}]}(t) \right| &\leq \kappa(k, R)(1 + \delta^{-4})(4t)^{\Re(1-a-b)/2} e^{-1/(32t)} \int_{1/\sqrt{4t}}^\infty x^{\Re(1-a-b)} e^{-x^2/8} dx \cdot \int_0^{\frac{1}{2}} \eta^{\delta-1} d\eta \\ &\leq \kappa(k, R) \mathcal{E}_N(\delta, R, t). \end{aligned}$$

To examine $\mathcal{H}_{a,b,k}^{[\frac{1}{2},1]}$, we again set $\tilde{x} = x\eta$ to express

$$\mathcal{H}_{a,b,k}^{[\frac{1}{2},1]}(t) = \frac{1}{\sqrt{4\pi t}} \int_1^\infty \int_{\frac{1}{2}}^1 x^{1-a-b} G_k(\eta; a, b) e^{-x^2(1-\eta)^2/(4t)} d\eta dx.$$

We expand $G_k(\eta; a, b)$ in a Taylor series about $\eta = 1$ on $[\frac{1}{2}, 1]$. Since $G_k(\eta; a, b) = O((1-\eta)^{2k+2})$ the first $2k+1$ terms vanish:

$$G_k(\eta; a, b) = \sum_{n=2k+2}^N \theta_{n,k}(a, b)(1-\eta)^n + r_N(\eta; a, b),$$

where $N \geq 2k+2$ and $|r_N(\eta; a, b)| \leq \kappa(k, N, R)(1 + \delta^{-4})(1-\eta)^{N+1}$. We may then expand:

$$\mathcal{H}_{a,b,k}^{[\frac{1}{2},1]}(t) = \sum_{n=2k+2}^N \frac{\theta_{n,k}(a, b)}{\sqrt{4\pi t}} \int_1^\infty \int_{\frac{1}{2}}^1 x^{1-a-b} (1-\eta)^n e^{-x^2(1-\eta)^2/(4t)} d\eta dx \quad (3.n)$$

$$+ \frac{1}{\sqrt{4\pi t}} \int_1^\infty \int_{\frac{1}{2}}^1 x^{1-a-b} r_N(\eta; a, b) e^{-x^2(1-\eta)^2/(4t)} d\eta dx. \quad (3.o)$$

Since $(a, b) \in \mathcal{O}_k$, $\Re(a + b) > -2k - 1$. Consequently $1 - \Re(a + b) < 2k + 2 \leq N$. This shows that $|x^{1-a-b}| \leq x^N$ for $x \in [1, \infty)$. We may therefore the remainder given in Equation (3.o) by:

$$\begin{aligned} & \frac{1}{\sqrt{4\pi t}} \int_1^\infty \int_{\frac{1}{2}}^1 \left| x^{1-a-b} r_N(\eta; a, b) e^{-x^2(1-\eta)^2/(4t)} \right| d\eta dx \\ & \leq \frac{\kappa(k, N, R)}{\sqrt{4\pi t}} (1 + \delta^{-4}) \int_1^\infty \int_{\frac{1}{2}}^1 x^N (1-\eta)^{N+1} e^{-x^2(1-\eta)^2/(4t)} d\eta dx \\ & = \frac{\kappa(k, N, R)}{\sqrt{4\pi t}} (1 + \delta^{-4}) \int_{\frac{1}{2}}^1 \int_1^\infty x^N (1-\eta)^{N+1} e^{-x^2(1-\eta)^2/(4t)} dx d\eta \\ & \leq \kappa(k, N, R) (1 + \delta^{-4}) t^{N/2} \int_{\frac{1}{2}}^1 \int_0^\infty \xi^{(N-1)/2} e^{-\xi} d\xi d\eta, \end{aligned}$$

where we have made the change of variables $x = \xi^{1/2}(1-\eta)^{-1}\sqrt{4t}$. The integral converges and leads to an estimate which is bounded by $\mathcal{E}_N(\delta, R, t)$.

Fix $n \geq 2k + 2$. We examine a typical term in the summation of Equation (3.n). We set $u = x(1-\eta)/\sqrt{4t}$ to express

$$\begin{aligned} & \frac{\theta_{n,k}(a, b)}{\sqrt{4\pi t}} \int_1^\infty \int_{\frac{1}{2}}^1 x^{1-a-b} (1-\eta)^n e^{-x^2(1-\eta)^2/(4t)} d\eta dx \\ & = \frac{\theta_{n,k}(a, b)}{\sqrt{\pi}} (4t)^{n/2} \int_1^\infty \int_0^{(1-\frac{1}{2})x/\sqrt{4t}} x^{-a-b-n} u^n e^{-u^2} du dx. \end{aligned}$$

The dx integral is convergent at ∞ since

$$\Re(a + b) \geq -2k - 1 + \delta \text{ implies } \Re(-a - b - n) \leq 2k + 1 - (2k + 2) - \delta = -1 - \delta.$$

Replacing the integral $0 \leq u \leq (1 - \frac{1}{2})x/\sqrt{4t}$ by $0 \leq u \leq \infty$ introduces an exponentially small error as $t \downarrow 0$ since $x \geq 1$. The existence of a series in the desired form now follows.

If $k = -1$, there are no regularizing terms and $G_{-1}(\eta; a, b) = \eta^{-a} + \eta^{-a}$. By Lemma 3.9, $C_{-1}(a, b) = c(a, b)$. Furthermore, expanding G_{-1} in a Taylor series about $\eta = 1$ shows that the coefficient of $t^{n/2}$ has the desired form since

$$\theta_{n,k}(a, b) = \frac{\theta_n(a) + \theta_n(b)}{n!} \quad \square$$

3.7. The proof of Theorem 3.1. Let $(a, b) \in \mathcal{O}_k$ for $k = -1, 0, 1, 2, 3$. We may expand

$$h_{a,b}^{[0,1]}(t) = h_{a,b,k}^{[0,1]}(t) + \sum_{\ell=0}^k \sigma_{k,\ell}(a, b) h_{a+b+\ell, -\ell}^{[0,1]}(t).$$

By Lemma 3.10, $h_{a,b,k}^{[0,1]}(t)$ has an appropriate asymptotic expansion. By Lemma 3.7, $h_{a+b+\ell, -\ell}^{[0,1]}(t)$ also has an appropriate asymptotic expansion for $\ell = 0, \dots, k$. Thus

$$\left| h_{a,b}^{[0,1]}(t) - \left\{ c_k(a, b) t^{(1-a-b)/2} + \sum_{n=0}^N c_{k,n}(a, b) t^{n/2} \right\} \right| \leq \mathcal{E}_N(\delta, R, t),$$

where the functions $c_k(a, b)$ and $c_{k,n}(a, b)$ are holomorphic on \mathcal{O}_k . Since $\mathcal{O}_i \cap \mathcal{O}_{i+1}$ is non-empty for $i = 0, 1, 2, 3$, we may use the identity theorem to conclude that $c_i(a, b) = c_{i+1}(a, b)$ and $c_{i,n} = c_{i+1,n}$ for $i = 0, 1, 2, 3$ and thus the holomorphic functions patch together properly on $\mathcal{O}_0 \cup \mathcal{O}_1 \cup \mathcal{O}_2 \cup \mathcal{O}_3$.

This yields expansions:

$$\begin{aligned} \left| h_{a,b}^{[0,1]}(t) - \left\{ c_{-1}(a,b)t^{(1-a-b)/2} + \sum_{n=0}^N c_{-1,n}(a,b)t^{n/2} \right\} \right| &\leq \mathcal{E}_N(\delta, R, t) \text{ on } \mathcal{O}_{-1}, \\ \left| h_{a,b}^{[0,1]}(t) - \left\{ c_0(a,b)t^{(1-a-b)/2} + \sum_{n=0}^N c_{0,n}(a,b)t^{n/2} \right\} \right| &\leq \mathcal{E}_N(\delta, R, t) \text{ on } \mathcal{U}, \text{ where} \\ \mathcal{U} := \mathcal{O}_0 \cup \mathcal{O}_1 \cup \mathcal{O}_2 = \{(a,b) : \Re(a) < 1, \Re(b) < 1, -7 < \Re(a+b) < 1, a+b \neq -1\}. \end{aligned}$$

We may now use Lemma 3.5 to relate $h_{a,b}^{[0,1]}$, $h_{a-2,b}^{[0,1]}$, and $h_{a-1,b-1}^{[0,1]}$. This shifts $\mathcal{O}_{-1} \cup \mathcal{O}_0$ (which is a disconnected open set) down into $\mathcal{O}_0 \cup \mathcal{O}_1$ (which is a connected open set). This permits us to extend $\{c_{-1}, c_{-1,n}\}$ and $\{c_0(a,b), c_{0,n}\}$ across dividing hyperplane $\Re(a+b) = 1$ and to remove the singularity except at $a+b = 1$. We denote the resulting extensions by $\{c_{-1}, c_{-1,n}\}$ and have an asymptotic series:

$$\left| h_{a,b}^{[0,1]}(t) - \left\{ c_{-1}(a,b)t^{(1-a-b)/2} + \sum_{n=0}^N c_{-1,n}(a,b)t^{n/2} \right\} \right| \leq \mathcal{E}_N(\delta, R, t) \text{ for} \\ \Re(a) < 1, \Re(b) < 1, a+b \neq 1, a+b \neq -1, -3 < \Re(a+b) < 2.$$

If $\Re(a) < 1$, if $\Re(b) < 1$, and if $\Re(a+b) > 1$, then the desired formulas $c_{-1}(a,b) = c(a,b)$ and $c_{-1,n}(a,b) = c_n(a,b)$ follow from Lemma 3.10. The formulas in general now follow by analytic continuation since we have extended c_{-1} and $c_{-1,n}$ holomorphically across the barrier $\Re(a+b) \neq -1$ omitting only the values $a+b = -1$. This proves Assertion (1) of Theorem 3.1 by giving a uniform estimate for the remainder if $(a,b) \in \mathcal{O}$ and $\Re(a+b) > -7$.

We use Lemma 3.4 to establish Assertion (2) of Theorem 3.1. We need to introduce cut-off functions. Let Ξ satisfy Equation (3.e). Let $\Re(a) < 1$, $\Re(b) < 1$, and $a+b \neq 1, -1, -3, \dots$. We may assume $\Re(a+b) \leq -7$. Let $\phi(x) := x^{-a}\Xi(x)$ and $\rho(x) := x^{-b}\Xi(x)$. We define heat content functions for $[0, 1]$ in $M = \mathbb{S}^1$ or in $M = \mathbb{R}$:

$$\begin{aligned} \beta_{a,b,\Xi}(M, t) &:= \beta_{[0,1]}(x^{-a}, x^{-b}, D_M), \\ \beta_{a,b,\Xi;1}(M, t) &:= \beta_{[0,1]}(\Xi x^{-a}, \Xi x^{-b}, D_M)(t), \\ \beta_{a,b,\Xi;2}(M, t) &:= \beta_{[0,1]}((1-\Xi)x^{-a}, \Xi x^{-b}, D_M)(t), \\ \beta_{a,b,\Xi;3}(M, t) &:= \beta_{[0,1]}(\Xi x^{-a}, (1-\Xi)x^{-b}, D_M)(t), \\ \beta_{a,b,\Xi;4}(M, t) &:= \beta_{[0,1]}((1-\Xi)x^{-a}, (1-\Xi)x^{-b}, D_M)(t). \end{aligned}$$

We may choose the notation so that $\Re(a) \leq \Re(b)$ and hence $\Re(a) \leq -\frac{7}{2}$. This implies $\Re(a) < 0$, $\Re(a+1) < 0$, and $\Re(a+2) < 0$. We now use Lemma 3.3 to obtain the following inequalities for $1 \leq i \leq 4$:

$$|\beta_{\tilde{a},b,\Xi;i}(S^1, t) - \beta_{a,b,\Xi;i}(\mathbb{R}, t)| \leq \kappa(R) \frac{1}{1 - \Re(b)} e^{-1/(8t)} \text{ for } \tilde{a} = a, a+1, a+2.$$

Since $(1-\Xi(x))x^{-a}$ is smooth on $[0, 1]$, Theorem 1.4 provides the existence of a suitable asymptotic series in $t^{n/2}$ for $\beta_{a,b,\Xi;2}(S^1, t)$ and $\beta_{a,b,\Xi;4}(S^1, t)$. Similarly, since $(1-\Xi(x))x^{-b}$ is smooth on $[0, 1]$, we may conclude that $\beta_{a,b,\Xi;3}(S^1, t)$ has a suitable asymptotic series as well. Consequently $\beta_{a,b}(S^1, t)$ has a suitable asymptotic series if and only if $\beta_{a,b,\Xi;1}(S^1, t)$ has such a series. Since $\Xi(x)x^{-a} \in C_0^1[0, 1]$, we may apply Lemma 3.4 (2),

$$\beta_{a,b,\Xi;1}(t) = \beta_{a,b,\Xi;1}(0) + \int_0^t \beta_{[0,1]}((\Xi x^{-a})'', \Xi x^{-b}, D_{S^1})(s) ds.$$

We expand

$$(\Xi(x)x^{-a})'' = a(a+1)\Xi(x)x^{-a-2} - 2a\Xi'(x)x^{-a-1} + \Xi''(x)x^{-a}.$$

By induction, we will have an asymptotic series for $\phi(x) = \Xi(x)x^{-a-2}$; the existence of a suitable series for $\phi(x) = \Xi'(x)x^{-a-1}$ or $\phi(x) = \Xi''(x)x^{-a}$ follows from Theorem 1.2 since these functions are smooth and are compactly supported in $(0, 1)$. (It is at that stage that our estimates become non-uniform in (a, b) since we did not take care in this regard in the proof of Theorem 1.2. This

defect could be easily remedied but it does not seem worth the additional technical fuss involved). We integrate the asymptotic series to establish Theorem 3.1 in complete generality; again, analytic continuation yields the values of the coefficients. This completes the proof of Theorem 3.1.

The following is a corollary to the arguments given above:

Corollary 3.11. *Let $(a, b) \in \mathcal{O}$ and let Ξ be a cut-off function satisfying Equation (3.e). There exist functions $c_{a,b,\Xi}$ and $\tilde{c}_{a,b,j,\Xi}$ which are holomorphic on \mathcal{O} so that if $N > \Re(-a-b) + 3$, then:*

$$\beta_{[0,1]}(\Xi x^{-a}, \Xi x^{-b}, D_{\mathbb{R}})(t) = \sum_{n=0}^N c_{a,b,n,\Xi} t^{n/2} + c(a,b) t^{(1-a-b)/2} + O(t^{(N+1)/2}) \text{ as } t \downarrow 0.$$

3.8. The log terms when $a+b=1$. We now use the uniform estimates of 3.1 near the hyperplane $\Re(a+b)=1$. We must restrict to $(a,b) \in \mathbb{R}^2$ since we will use certain monotonicity results that fail if a and b can be complex.

Equation (3.b) shows that c_0 and $c(a,b)$ have simple poles when $a+b=1$. This gives rise to a logarithmic term in the asymptotic expansion at these values when the associated terms in the asymptotic expansion collide. We begin with a technical result:

Lemma 3.12. *Let $\mathcal{X}(a,b) := (1-a-b)c(a,b)$.*

- (1) $\mathcal{X}(a,b)$ extends holomorphically to the plane $a+b=1$ by setting $\mathcal{X}(a,1-a) = -1$.
- (2) Let $\mathcal{Y}(a) := \partial_{\delta}\{\mathcal{X}(a+\delta,1-a)\}|_{\delta=0}$. Let $k \geq 3$. If $a \in [\epsilon, 1-\epsilon]$ for some $\epsilon > 0$, if $|\delta| < \min\{\epsilon/2, t\}$, and if $t \leq t_0(\epsilon)$, then:

$$\left| \left\{ c(a+\delta,1-a) t^{-\delta/2} - \frac{1}{\delta} \right\} - \left\{ -\frac{1}{2} \ln t + \mathcal{Y}(a) \right\} \right| \leq C(\epsilon) \delta (1 + \ln(t)^2).$$

Proof. We use the recursion relation $(1-a-b)\Gamma(a+b-1) = -\Gamma(a+b)$ to express

$$\mathcal{X}(a,b) = -2^{-a-b} \frac{1}{\sqrt{\pi}} \Gamma\left(\frac{2-a-b}{2}\right) \Gamma(a+b) \cdot \left(\frac{\Gamma(1-a)}{\Gamma(b)} + \frac{\Gamma(1-b)}{\Gamma(a)} \right).$$

This is no longer singular when $a+b=1$ and we have $\mathcal{X}(a,1-a) = -1$. This verifies Assertion (1). We have by definition that

$$\begin{aligned} c(a+\delta,1-a) t^{-\delta/2} - \frac{1}{\delta} &= - \left\{ \frac{\mathcal{X}(a+\delta,1-a) t^{-\delta/2} - \mathcal{X}(a,1-a) t^0}{\delta} \right\} \\ &= -\mathcal{X}(a+\delta,1-a) \left\{ \frac{t^{-\delta/2} - 1}{\delta} \right\} - \left\{ \frac{\mathcal{X}(a+\delta,1-a) - \mathcal{X}(a,1-a)}{\delta} \right\}. \end{aligned}$$

We have by definition that

$$- \left\{ \frac{\mathcal{X}(a+\delta,1-a) - \mathcal{X}(a,1-a)}{\delta} \right\} = -\mathcal{Y}(a) + O(\delta)$$

where the remainder may be estimated uniformly if δ is small and if a has a compact parameter range. The desired estimate for this error follows. Since $-\mathcal{X}(a+\delta,1-a) = 1 + O(\delta)$, it suffices to examine

$$\frac{e^{-\frac{\delta}{2} \ln(t)} - 1}{\delta} = \sum_{n=1}^{\infty} \frac{1}{n!} \left(-\frac{1}{2} \ln(t) \right)^n \delta^{n-1} = -\frac{1}{2} \ln(t) - \left(\frac{1}{2} \ln(t) \right)^2 \delta \sum_{n=2}^{\infty} \frac{(-\frac{1}{2} \ln(t))^{n-2}}{n!}.$$

Since $|\delta| < t$, we can choose t sufficiently small so that $|\delta \ln(t)| < |t \ln t| < 1$ so the infinite series can be uniformly bounded by e . This proves Assertion (2). \square

Lemma 3.13. *If $a+b=1$, $a \in (0,1)$, if $b \in (0,1)$, and if $k \geq 2$, then*

$$\left| h_{a,1-a}^{[0,1]}(t) - \left\{ -\frac{1}{2} \ln t + \mathcal{Y}(a) + \sum_{n=1}^k c_n(a,1-a) t^{n/2} \right\} \right| \leq O(t^{(k+1)/2}).$$

Proof. Let $k \geq 2$ be given. Choose N so that $\frac{N}{10} \geq 2 + \frac{k+1}{2}$. We first establish the estimate:

$$\left| h_{a,1-a}^{[0,1]}(t) - \left\{ -\frac{1}{2} \ln t + \mathcal{Y}(a) + \sum_{n=1}^N c_n(a, 1-a) t^{n/2} \right\} \right| = O(t^{N/10} \ln(t)^2). \quad (3.p)$$

We use Theorem 3.1 to see

$$h_{a+\delta,1-a}^{[0,1]}(t) = c(a+\delta, 1-a) t^{-\delta/2} + \frac{1}{1-a-b} + \dots$$

Let $a \in [\epsilon, 1-\epsilon]$ and let $0 < \delta < \epsilon/2$. We estimate from above:

$$\begin{aligned} & h_{a,1-a}^{[0,1]}(t) - \sum_{n=1}^N c_n(a, 1-a) t^{n/2} + \frac{1}{2} \ln t - \mathcal{Y}(a) \\ \leq & h_{a+\delta,1-a}^{[0,1]}(t) - \sum_{n=1}^N c_n(a, 1-a) t^{n/2} + \frac{1}{2} \ln t - \mathcal{Y}(a) \\ = & h_{a+\delta,1-a}^{[0,1]}(t) - \sum_{n=0}^N c_n(a+\delta, 1-a) t^{n/2} + c(a+\delta, 1-a) t^{-\delta/2} \end{aligned} \quad (3.q)$$

$$+ \sum_{n=1}^N \{c_n(a+\delta, 1-a) - c_n(a, 1-a)\} t^{n/2} \quad (3.r)$$

$$+ \left\{ \frac{1}{2} \ln t - \mathcal{Y}(a) + c(a+\delta, 1-a) t^{-\delta/2} + \frac{1}{\delta} \right\} \quad (3.s)$$

We take $\delta = t^\gamma$. We use Theorem 3.1 to estimate the terms in Equation (3.q) by

$$C(N, \epsilon)(1 + t^{-4\gamma})(t^{N/2} + e^{-1/(36t)}) \leq C(N, \epsilon) t^{-4\gamma+N/2} \text{ for } t \leq t_0(N, \epsilon).$$

The terms c_n for $1 \leq n \leq N$ are regular at $a+b=1$. Thus we may estimate

$$|c_n(a+\delta, 1-a) - c_n(a, 1-a)| \leq C(N, \epsilon) \delta = C(N, \epsilon) t^\gamma \text{ for } 1 \leq n \leq N,$$

and thereby control the terms in Equation (3.r). In order to control the terms in Equation (3.s) we use Lemma 3.12, and henceforth require $\gamma \geq 1$. Combining all three remainders provides an upper estimate

$$C(\epsilon) t^\gamma \ln(t)^2 + C(N, \epsilon) t^{-4\gamma+N/2} + C(N, \epsilon) t^\gamma.$$

We now choose $\gamma = N/10$ as to minimize this upper estimate. This gives for $N \geq 10$, $O(t^{N/10} \ln(t)^2)$. Reversing the sign of δ provides the desired lower estimate and establishes Equation (3.p).

Since $t \log(t)$ tends to 0 as $t \rightarrow 0$ and since $N/10 \geq \frac{k+1}{2} + 2$,

$$t^{N/10} \ln(t)^2 \leq t^{\frac{k+1}{2}} \quad \text{as } t \downarrow 0$$

and consequently Equation (3.p) yields:

$$\left| h_{a,1-a}^{[0,1]}(t) - \left\{ -\frac{1}{2} \ln t + \mathcal{Y}(a) + \sum_{n=1}^k c_n(a, 1-a) t^{n/2} + \sum_{n=k+1}^N c_n(a, 1-a) t^{n/2} \right\} \right| = O(t^{\frac{k+1}{2}}). \quad (3.t)$$

The Lemma now follows since

$$\left| \sum_{n=k+1}^N c_n(a, 1-a) t^{n/2} \right| = O(t^{\frac{k+1}{2}}). \quad \square$$

3.9. The proof of Theorem 3.2. Theorem 3.2 follows from Lemma 3.13 if $a + b = 1$. Suppose first $a + b = -1$ so $k = 1$. Let $\phi(x) = x^{-a}\Xi_1(x)$ and let $\rho(x) = x^{-b}\Xi_2(x)$. We may suppose $a \leq b$. There are 3 cases:

(1) $0 < b < 1$. As $a < -1$, we may apply Lemma 3.4 (2) to see:

$$\begin{aligned}\beta_{[0,1]}(\phi, \rho, D_{S^1})(t) &= \beta_{[0,1]}(\phi, \rho, D_{S^1})(0) + \int_0^t \beta(\phi'', \rho, D_{S^1})(s) ds \\ &= \beta_{[0,1]}(\phi, \rho, D_{S^1})(0) + a(a+1) \int_0^t \beta_{[0,1]}(x^{-(a+2)}\Xi_1, x^{-b}\Xi_2, D_{S^1})(s) ds \\ &\quad + \int_0^t \beta_{[0,1]}(-2ax^{-a-1}\Xi'_1 + x^{-a}\Xi'', x^{-b}\Xi_2, D_{S^1})(s) ds.\end{aligned}$$

Since $\{-2ax^{-a-1}\Xi'_1 + x^{-a}\Xi''\}$ is smooth compactly supported in $(0, 1)$, we use Theorem 1.3 to obtain a suitable asymptotic series for $\beta(-2ax^{-a-1}\Xi'_1 + x^{-a}\Xi'', x^{-b}\Xi_2, D_{S^1})(s)$ which can be then integrated term by term to get an appropriate full asymptotic series in t . We integrate the asymptotic series of Theorem 1.2 for $(a+2) + b = 1$ for $\beta(x^{-(a+2)}\Xi_1, x^{-b}\Xi_2, D_{S^1})(s)$ to establish the desired series in the setting:

$$\beta_{[0,1]}(\phi, \rho, D_{S^1})(t) = \beta_{0,1} + \beta_{1,1}t - \frac{a(a+1)}{2}t \log(t) + \dots$$

(2) $b < 0$ and $a < 0$. We apply Lemma 3.4 (3) to see

$$\beta_{[0,1]}(\phi, \rho, D_{S^1})(t) = \beta_{0,1} + \beta_{1,1}t + \frac{ab}{2}t \log(t) + \dots$$

Note the sign change. We now set $b = -1 - a$ to see the coefficient of $t \log t$ is $-\frac{a(a+1)}{2}$ as desired.

(3) $b = 0$ and $a = -1$. In this case the argument given above fails since Lemma 3.4 is not applicable. Instead we use Theorem 1.4 to conclude there are no log terms and we have a full asymptotic series; the coefficient $a(a+1) = 0$ in this setting.

We now proceed by induction and suppose that an appropriate asymptotic series has been established if $\tilde{a} + \tilde{b} = 1 - 2k \leq -1$. Suppose that $a + b = 1 - 2k - 2 \leq -3$. We suppose that $a \leq b$ and thus $2a \leq -3$ so $a \leq -3/2$. We apply the argument above and use the existence of the asymptotic series for $(a+2, b)$. The coefficient of $t^k \log(t)$ in $\beta_{[0,1]}(\phi'', \rho, D_{S^1})(t)$ is

$$-\frac{(a+2)(a+3)\dots(a+2+2k-1)}{2 \cdot k!}$$

and thus the coefficient of $t^{k+1} \log(t)$ in $\beta_{[0,1]}(\phi, \rho, D_{S^1})(t)$ is

$$-\frac{(a+2)((a+2)+1)\dots((a+2)+2k-1)}{2 \cdot k!} \cdot \frac{a(a+1)}{k+1} = -\frac{a(a+1)\dots(a+2(k+1)-1)}{2 \cdot (k+1)!}.$$

The desired result now follows. \square

Again, the following is an immediate consequence of the arguments we have given above.

Lemma 3.14. *Let $(a, b) \in \mathbb{R}^2$ satisfy $\Re(a) < 1$, $\Re(b) < 1$, and $a + b = 1 - 2k$ where k is a non-negative integer. Let Ξ be a cut-off function satisfying Equation (3.e). There exist functions $c_{a,b,n,\Xi}$ so that if $N > -\Re(a+b) + 3$, then*

$$\beta_{[0,1]}(\Xi x^{-a}, \Xi x^{-b}, D_{\mathbb{R}})(t) = \sum_{n=0}^N c_{a,b,n,\Xi} t^{n/2} - \frac{a(a+1)\dots(a+2(k+1)-1)}{2 \cdot (k+1)!} \log(t) t^k + O(t^{(N+1)/2})$$

as $t \downarrow 0$.

4. THE PSEUDO-DIFFERENTIAL CALCULUS

In Section 4, we will use the calculus of pseudo-differential to complete the proof of Theorem 1.1; the special case computation of Theorem 3.1 is an essential ingredient. Throughout this section, we assume the ambient Riemannian manifold (M, g) is compact and without boundary, and that Ω is a compact smooth subdomain of M with smooth boundary $\partial\Omega$. We assume that $(a, b) \in \mathcal{O}$ where \mathcal{O} is as defined in Equation (1.b). We assume that ϕ and ρ are smooth on the interior of Ω and that $r^a\phi$ and $r^b\phi$ are smooth near the boundary of Ω .

By using a partition of unity, we may assume that ρ and ϕ are supported within coordinate systems. Since the kernel of the heat equation decays exponentially in t for $\text{dist}_g(x, \tilde{x}) \geq \epsilon > 0$, we may assume that ρ and ϕ have support within the same coordinate system. There will, of course, be three different types of coordinate systems to be considered - those which touch the boundary of Ω , those which are contained entirely within the interior of Ω , and those which are contained in the exterior of Ω ; those contained in the exterior of Ω play no role as they contribute an exponentially small error as $t \downarrow 0$. In Section 4.1 we establish notational conventions and prove a technical result. In Section 4.2 we use the pseudo-differential calculus to construct an approximation to the resolvent to the kernel of the heat equation that we use to begin our study of the heat content. In Section 4.3, we shall examine coordinate systems contained in the interior of Ω . We complete the proof of Theorem 1.1 in Section 4.4.

4.1. Notational conventions. Let $x = (x^1, \dots, x^m) \in \mathbb{R}^m$ be coordinates on an open set $\mathcal{U} \subset M$. Let (x, ξ) be the induced coordinate system on the cotangent space $T^*(\mathcal{U})$ where we expand a 1-form $\omega \in T^*(\mathcal{U})$ in the form $\omega = \xi_i dx^i$ to define the dual coordinates $\xi = (\xi_1, \dots, \xi_m)$. We let $x \cdot \xi$ be the natural Euclidean pairing $x \cdot \xi := x^i \xi_i$. If $\alpha = (a_1, \dots, a_m)$ is a multi-index, set

$$\begin{aligned} |\alpha| &= a_1 + \dots + a_m, & \alpha! &= a_1! \dots a_m!, & \partial_x^\alpha &:= \partial_{x_1}^{a_1} \dots \partial_{x_m}^{a_m}, \\ \phi^{(\alpha)} &:= \partial_x^\alpha \phi, & \rho^{(\alpha)} &:= \partial_x^\alpha \rho, & D_x^\alpha &:= \sqrt{-1}^{|\alpha|} d_x^\alpha, \\ d_\xi^\alpha &:= \partial_{\xi_1}^{a_1} \dots \partial_{\xi_m}^{a_m}, & x^\alpha &= x_1^{a_1} \dots x_m^{a_m}, & \xi^\alpha &:= \xi_1^{a_1} \dots \xi_m^{a_m}. \end{aligned}$$

For example, with these notational conventions, Taylor's theorem becomes

$$f(x) = \sum_{|\alpha| \leq n} \frac{1}{\alpha!} f^{(\alpha)}(x_0) (x - x_0)^\alpha + O(|x - x_0|^{n+1}). \quad (4.a)$$

Let dv_x , $dv_{\tilde{x}}$, and dv_ξ be the usual Euclidean Lebesgue measure on \mathbb{R}^m . Let L_e^2 denote $L^2(\mathbb{R}^m)$ with respect to Lebesgue measure. Let $g(x) = g_{ij}(x) dx^i \circ dx^j$ be a Riemannian metric on \mathcal{U} and define:

$$\|\xi\|_{g^*(x)}^2 := g^{ij}(x) \xi_i \xi_j \text{ and } \|x - \tilde{x}\|_{g(x)}^2 := g_{ij}(x) (x^i - \tilde{x}^i) (x^j - \tilde{x}^j).$$

We shall always be restricting to compact x and \tilde{x} subsets. Let $(\cdot, \cdot)_e$ and $\|\cdot\|_e$ denote the usual Euclidean inner product and norm, respectively:

$$(x, y)_e := x_1 y_1 + \dots + x_m y_m \text{ and } \|(x_1, \dots, x_m)\|_e^2 := (x, x)_e = x_1^2 + \dots + x_m^2.$$

We have estimates:

$$C_1 \|\xi\|_e^2 \leq \|\xi\|_{g^*(x)}^2 \leq C_2 \|\xi\|_e^2 \text{ and } C_1 \|x - \tilde{x}\|_e^2 \leq \|x - \tilde{x}\|_{g(x)}^2 \leq C_2 \|x - \tilde{x}\|_e^2$$

for positive constants C_i . Let $\theta = \sqrt{g}$; θ is a symmetric metric so that $\theta_{ij} \theta_{jk} = g_{ik}$. We then have

$$\|\xi\|_{g^*(x)}^2 = \|\theta^{-1}(x) \xi\|_e^2 \text{ and } \|(x - \tilde{x})\|_{g(x)}^2 = \|\theta(x) (x - \tilde{x})\|_e^2.$$

Use $\theta^2 = g$ to lower indices and regard $\theta^2(x) (x - \tilde{x})$ as a vector $(X - \tilde{X})_i := g_{ij}(x) (x^j - \tilde{x}^j)$.

Lemma 4.1. $\|\xi\|_{g^*(x)}^2 + \sqrt{-1} (x - \tilde{x}) \cdot \xi / \sqrt{t} = \|\xi + \frac{1}{2} \sqrt{-1} (X - \tilde{X}) / \sqrt{t}\|_{g^*(x)}^2 + \|(x - \tilde{x})\|_{g(x)}^2 / (4t).$

Proof. We expand:

$$\begin{aligned} & \|\xi\|_{g^*(x)}^2 + \sqrt{-1} (x - \tilde{x}) \cdot \xi / \sqrt{t} \\ &= (\theta^{-1}(x) \xi, \theta^{-1}(x) \xi)_e + \sqrt{-1} (\theta^{-1}(x) \theta^2(x) (\tilde{x} - x) / \sqrt{t}, \theta^{-1}(x) \xi)_e \end{aligned}$$

$$\begin{aligned}
&= (\theta^{-1}(x)\{\xi + \sqrt{-1}\theta^2(x)(\tilde{x} - x)/\sqrt{t}\}, \theta^{-1}(x)\xi)_e \\
&= \|\theta^{-1}(x)\{\xi + \frac{1}{2}\sqrt{-1}\theta^2(x)(\tilde{x} - x)/\sqrt{t}\}\|_e^2 + \|\theta(x)(\tilde{x} - x)\|_e^2/(4t) \\
&= \|\xi + \frac{1}{2}\sqrt{-1}(X - \tilde{X})/\sqrt{t}\|_{g^*(x)}^2 + \|(x - \tilde{x})\|_{g(x)}^2/(4t). \quad \square
\end{aligned}$$

4.2. The resolvent and the heat content. If P is a pseudo-differential operator with symbol $p(x, \xi)$, then P is characterized by following identity for all $\phi \in C_0^\infty(V)$ and $\rho \in C_0^\infty(V^*)$:

$$\langle P\phi, \rho \rangle_{L_e^2} = (2\pi)^{-m} \iiint e^{-\sqrt{-1}(x-\tilde{x})\cdot\xi} \langle p(x, \xi)\phi(x), \rho(\tilde{x}) \rangle d\nu_x d\nu_\xi d\nu_{\tilde{x}}. \quad (4.b)$$

The integrals in question here are iterated integrals - the convergence is not absolute and the $d\nu_x$ integral has to be performed before the $d\nu_\xi$ integral. However, if $p(x, \xi)$ decays rapidly enough in ξ , then the integrals are in fact absolutely convergent and we can interchange the order of integration to see following [11, Lemma 1.2.5] that P is given by a kernel:

$$\begin{aligned}
\langle P\phi, \rho \rangle_{L_e^2} &= \int \langle K_P(x, \tilde{x})\phi(x), \rho(\tilde{x}) \rangle d\nu_x d\nu_{\tilde{x}}, \text{ where} \\
K_P(x, \tilde{x}) &:= (2\pi)^{-m} \int e^{-\sqrt{-1}(x-\tilde{x})\cdot\xi} p(x, \xi) d\nu_\xi. \quad (4.c)
\end{aligned}$$

Let D_M be an operator of Laplace type on $C^\infty(V)$ over M . This means that in a system of local coordinates (x^1, \dots, x^m) on an open subset \mathcal{U} of M and relative to a local frame for V that D has the form given in Equation (1.a), i.e. after changing notation slightly that we have:

$$D_M = a_2^{ij}(x)D_{x_i}D_{x_j} + a_1^i(x)D_{x_i} + a_0(x).$$

We ensure that Equation (4.b) defines the operator D_M by defining:

$$\begin{aligned}
p(x, \xi) &= p_2(x, \xi) + p_1(x, \xi) + p_0(x, \xi), \text{ where} \\
p_2(x, \xi) &:= |\xi|_{g^*(x)}^2, \quad p_1(x, \xi) := a_1^i(x)\xi_i, \quad p_0(x, \xi) := a_0(x). \quad (4.d)
\end{aligned}$$

Let $\mathcal{R} \subset \mathbb{C}$ be the complement of a cone of some small angle about the positive real axis and a ball of large radius about the origin where \mathcal{R} is chosen so that D_M has no eigenvalues in \mathcal{R} . We let $\lambda \in \mathcal{R}$ henceforth. Following the discussion of [11, Lemma 1.7.2], we define $r_n(x, \xi; \lambda)$ for $(x, \xi) \in T^*(\mathcal{U})$ and $\lambda \in \mathcal{R}$ inductively by setting:

$$\begin{aligned}
r_0(x, \xi; \lambda) &:= (|\xi|_{g^*(x)}^2 - \lambda)^{-1}, \\
r_n &:= -r_0 \sum_{|\alpha|+2+j-k=n, j < n} \frac{1}{\alpha!} \partial_\xi^\alpha p_k \cdot D_x^\alpha r_j \text{ for } n > 0. \quad (4.e)
\end{aligned}$$

Define

$$\begin{aligned}
\text{ord}(\partial_x^\alpha p_2) &= |\alpha|, \quad \text{ord}(\partial_x^\alpha p_1) = |\alpha| + 1, \quad \text{ord}(\partial_x^\alpha p_0) = |\alpha| + 2, \\
\text{weight}(\lambda) &= 2, \quad \text{weight}(\xi) = 1.
\end{aligned}$$

The following lemma follows immediately by induction from the recursive definition in Equation (4.e):

Lemma 4.2. r_n is homogeneous of order n in the derivatives of the symbol of D_M with weight $-n-2$ in (ξ, λ) . There exist polynomials $r_{n,j,\alpha}(x, D_M)$ for $n \leq j \leq 3n$ which are homogeneous of order n in the derivatives of the symbol of D_M so:

$$r_n(x, \xi; \lambda) = \sum_{2j-|\alpha|=n} r_{n,j,\alpha}(x, D_M) (|\xi|_{g^*(x)}^2 - \lambda)^{-j-1} \xi^\alpha.$$

We use Equation (4.b) to define the pseudo-differential operator $R_n(\lambda)$ with symbol r_n so that

$$\langle R_n(\lambda)\phi, \rho \rangle_{L_e^2} = (2\pi)^{-m} \iiint e^{-\sqrt{-1}(x-\tilde{x})\cdot\xi} \langle r_n(x, \xi; \lambda)\phi(x), \rho(\tilde{x}) \rangle d\nu_x d\nu_\xi d\nu_{\tilde{x}}.$$

Let $\|_{-k,k}$ be the norm of a map from the Sobolev space H_{-k} to the Sobolev space H_k . By [11, Lemma 1.7.3] we have that if $\lambda \geq \lambda(k)$ and if $n \geq n(k)$, then:

$$\|(D_M - \lambda)^{-1} - R_0(\lambda) - \dots - R_n(\lambda)\|_{-k,k} \leq C_k(1 + |\lambda|)^{-k}.$$

Orient the boundary γ of \mathcal{R} suitably. We use the operator valued Riemann integral to define

$$e^{-tD_M} := \frac{1}{2\pi\sqrt{-1}} \int_{\gamma} e^{-t\lambda} (D_M - \lambda)^{-1} d\lambda.$$

We use [11, Lemma 1.7.5] to see that this is the fundamental solution of the heat equation and belongs to $\text{Hom}(H_{-k}, H_k)$ for any k . We now let

$$e_n(x, \xi; t) = \frac{1}{2\pi\sqrt{-1}} \int_{\gamma} e^{-t\lambda} r_n(x, \xi; \lambda) d\lambda \quad (4.f)$$

define the pseudo-differential operator

$$E_n(t, D_M) := \frac{1}{2\pi\sqrt{-1}} \int_{\gamma} e^{-t\lambda} R_n(\lambda) d\lambda. \quad (4.g)$$

We use Lemma 4.2 (3) and Cauchy's integral formula to rewrite Equation (4.f) as:

$$e_n(x, \xi; t) = \sum_{2j-|\alpha|=n} \frac{t^j}{j!} \xi^\alpha e^{-t|\xi|_{g^*(x)}^2} r_{n,j,\alpha}(x, D_M). \quad (4.h)$$

By Equation (4.c) and Equation (4.h), the operator E_n of Equation (4.g) is given by the smooth kernel

$$K_n(x, \tilde{x}; t) := \sum_{2j-|\alpha|=n} (2\pi)^{-m} \frac{t^j}{j!} \int_{\mathbb{R}^m} e^{-t|\xi|_{g^*(x)}^2 - \sqrt{-1}(x-\tilde{x}) \cdot \xi} \xi^\alpha r_{n,j,\alpha}(x, D_M) d\nu_\xi. \quad (4.i)$$

Let $\|\cdot\|_{C^k}$ denote the C^k norm. Given any $k \in \mathbb{N}$, there exists $n(k)$ so that if $n \geq n(k)$ and if $0 < t < 1$, then [11, Lemma 1.8.1] implies:

$$\|e^{-tD_M} - \sum_{n=0}^{n(k)} E_n(t, D_M)\|_{-k,k} \leq C_k t^k$$

This gives rise to a corresponding estimate (after increasing $n(k)$ appropriately):

$$\|K(t, x, \tilde{x}, D_M) - \sum_{n=0}^{n(k)} K_n(t, x, \tilde{x}, D_M)\|_{C^k} \leq C_k t^k. \quad (4.j)$$

We use Equation (4.h), Equation (4.i), and Equation (4.j) to expand

$$\begin{aligned} \beta(\phi, \rho, D_M)(t) &= \sum_{2j-|\alpha|=0}^{n(k)} (2\pi)^{-m} \frac{t^j}{j!} \iiint e^{-t|\xi|_{g^*(x)}^2 - \sqrt{-1}(x-\tilde{x}) \cdot \xi} \xi^\alpha \\ &\quad \times \langle r_{n,j,\alpha}(x, D_M) \phi(x), \rho(\tilde{x}) \rangle d\nu_x d\nu_\xi d\nu_{\tilde{x}} + O(t^k). \end{aligned}$$

We examine a typical term in the sum setting:

$$\begin{aligned} \beta_{n,j,\alpha}(\phi, \rho)(t) &:= (2\pi)^{-m} \frac{t^j}{j!} \iiint e^{-t|\xi|_{g^*(x)}^2 - \sqrt{-1}(x-\tilde{x}) \cdot \xi} \xi^\alpha \\ &\quad \times \langle r_{n,j,\alpha}(x, D_M) \phi(x), \rho(\tilde{x}) \rangle d\nu_x d\nu_\xi d\nu_{\tilde{x}}. \end{aligned}$$

Here all integrals are over \mathbb{R}^m and converge absolutely for $t > 0$; ϕ and ρ have compact support. We change variables setting $\tilde{\xi} := t^{1/2}\xi$ to express:

$$\begin{aligned} \beta_{n,j,\alpha}(\phi, \rho)(t) &= \frac{t^{j-\frac{1}{2}m-\frac{1}{2}|\alpha|}}{j!} (2\pi)^{-m} \iiint e^{-|\tilde{\xi}|_{g^*(x)}^2 - \sqrt{-1}(x-\tilde{x}) \cdot \tilde{\xi}/\sqrt{t}} \tilde{\xi}^\alpha \\ &\quad \times \langle r_{n,j,\alpha}(x, D_M) \phi(x), \rho(\tilde{x}) \rangle d\nu_x d\nu_{\tilde{\xi}} d\nu_{\tilde{x}}. \end{aligned}$$

Note that $\frac{1}{2}n = j - \frac{1}{2}|\alpha|$. We adopt the notation of Lemma 4.1 and make a complex change of coordinates setting:

$$\eta = \tilde{\xi} + \frac{1}{2}\sqrt{-1}(X - \tilde{X})/\sqrt{t}.$$

We then apply Lemma 4.1 and the binomial theorem to express:

$$\begin{aligned} & \beta_{n,j,\alpha}(\phi, \rho)(t) \\ = & (2\pi)^{-m} \sum_{\alpha_1 + \alpha_2 = \alpha} \frac{\alpha!}{j! \alpha_1! \alpha_2!} (-\sqrt{-1})^{|\alpha_2|} t^{(n-m)/2} \iint e^{-\|\eta\|_{g(x)}^2} \eta^{\alpha_1} \\ & \times e^{-\|x-\tilde{x}\|_{g(x)}^2/(4t)} \left(\frac{X-\tilde{X}}{2\sqrt{t}} \right)^{\alpha_2} \langle r_{n,j,\alpha}(x, D_M) \phi(x), \rho(\tilde{x}) \rangle d\nu_\eta d\nu_x d\nu_{\tilde{x}}. \end{aligned}$$

The $d\nu_\eta$ integral is over the complex domain $\eta \in \mathbb{R} + \frac{1}{2}\sqrt{-1} \frac{X-\tilde{X}}{\sqrt{t}}$. But we can deform that domain back to the real domain $\eta \in \mathbb{R}$. Set

$$\begin{aligned} c_{\alpha_1, \alpha_2, j} &:= (2\pi)^{-m} \frac{1}{j!} \frac{(\alpha_1 + \alpha_2)!}{\alpha_1! \alpha_2!} (-\sqrt{-1})^{|\alpha_2|} \int \eta^{\alpha_1} e^{-\|\eta\|_{g^*(x)}^2} d\nu_\eta \text{ to express} \\ \beta_{n,j,\alpha}(t) &= \sum_{\alpha_1 + \alpha_2 = \alpha} c_{\alpha_1, \alpha_2, j} t^{(n-m)/2} \\ & \times \iint e^{-\|x-\tilde{x}\|_{g(x)}^2/(4t)} \left(\frac{X-\tilde{X}}{2\sqrt{t}} \right)^{\alpha_2} \langle r_{n,j,\alpha}(x, D_M) \phi(x), \rho(\tilde{x}) \rangle d\nu_x d\nu_{\tilde{x}}. \end{aligned}$$

This sum ranges over $|\alpha_1|$ even as otherwise c_{α_1, α_2} vanishes. Thus $|\alpha_2| \equiv |\alpha| \equiv n \pmod{2}$. This reduces the proof to considering expressions of the form:

$$\begin{aligned} f_{n,j,\alpha,\alpha_2}(t) &:= t^{(n-m)/2} \iint e^{-\|x-\tilde{x}\|_{g(x)}^2/(4t)} \left(\frac{X-\tilde{X}}{\sqrt{t}} \right)^{\alpha_2} \\ & \times \langle r_{n,j,\alpha}(x, D_M) \phi(x), \rho(\tilde{x}) \rangle d\nu_x d\nu_{\tilde{x}}, \\ & \text{where } |\alpha_2| \equiv n \pmod{2} \text{ and } \text{ord}(r_{n,j,\alpha}(x, D_M)) = n. \end{aligned} \tag{4.k}$$

In examining such integrals, the fact that we are in the vector valued setting plays no role and we assume henceforth that V and V^* are 1-dimensional and hence r , ϕ , and ρ are scalar valued.

4.3. The interior terms in Theorem 1.1. We begin our analysis with:

Lemma 4.3. *Let $\Phi \in L^1(\mathbb{R}^m)$, let $\rho \in C^k(\mathbb{R}^m)$ have compact support in an open subset $\mathcal{U} \subset \mathbb{R}^m$, and let $(X - \tilde{X})_i := g_{ij}(x - \tilde{x})^j$. Let*

$$F(t) := t^{(n-m)/2} \int_{\mathcal{U}} \int_{\mathcal{U}} e^{-\|x-\tilde{x}\|_{g(x)}^2/(4t)} \left(\frac{X-\tilde{X}}{\sqrt{t}} \right)^{\alpha_2} \langle \Phi(x), \rho(\tilde{x}) \rangle d\nu_x d\nu_{\tilde{x}}.$$

There exist smooth coefficients $c_{\sigma, \alpha_2, g}(x)$ so that as $t \downarrow 0^+$,

$$\left| F(t) - \sum_{|\sigma|=0}^{k-1} t^{(n+|\sigma|)/2} \int_{\mathcal{U}} c_{\sigma, \alpha_2, g}(x) \langle \Phi(x), \rho^{(\sigma)}(x) \rangle d\nu_x \right| \leq Ct^k.$$

Proof. We make the change of variables $\tilde{x} = x + u$ and dually $\tilde{X} = X + U$ where $U_i = g_{ij}u^j$ to express

$$F(t) = t^{(n-m)/2} \iint e^{-\|u\|_{g(x)}^2/(4t)} \left(\frac{U}{\sqrt{t}} \right)^{\alpha_2} \langle \Phi(x), \rho(x+u) \rangle d\nu_u d\nu_x.$$

The $d\nu_u$ integral decays exponentially for $|u| > t^{1/4}$ so we may assume the $d\nu_u$ integral is localized to $|u| < t^{1/4}$. For u small, we use Equation (4.a) to express:

$$\begin{aligned} \rho(x+u) &\sim \sum_{|\sigma|=0}^{k-1} \frac{1}{\sigma!} u^\sigma d_x^\sigma \rho(x) + O(|u|^k), \\ F(t) &= t^{(n-m)/2} \sum_{|\sigma| \leq k-1} \frac{1}{\sigma!} \iint e^{-\|u\|_{g(x)}^2/(4t)} \\ & \times \left\{ \left(\frac{U}{\sqrt{t}} \right)^{\alpha_2} u^\sigma \langle \Phi(x), \rho^{(\sigma)}(x) \rangle + O(|u|^k) \right\} d\nu_u d\nu_x. \end{aligned}$$

We set $\tilde{u} = u/\sqrt{t}$ and $\tilde{U} = U/\sqrt{t}$ to express

$$F(t) = \sum_{|\sigma|=0}^{k-1} \frac{1}{\sigma!} t^{(n+|\sigma|)/2} \iint e^{-\|\tilde{u}\|_{g(x)}^2/4} \tilde{U}^{\alpha_2} \tilde{u}^\sigma \langle \Phi(x), \rho^{(\sigma)}(x) \rangle d\nu_{\tilde{u}} d\nu_x + O(t^k).$$

The $d\nu_x$ integral remains an integral over \mathcal{U} . But as $t \downarrow 0$, the $d\nu_{\tilde{u}}$ integral expands to \mathbb{R}^m and defines the coefficients c_{σ, α_2} . \square

We apply Lemma 4.3 to the case $\Phi = r_{n,j,\alpha} \phi$ in Equation (4.k). By assumption $r_{n,j,\alpha}$ is of order n in the derivatives of the total symbol of D_M . We have $\rho^{(\sigma)}$ is of order $|\sigma|$ in the derivatives of ρ . Thus we have expressions which are of order $n + |\sigma|$ in the derivatives of the symbol of D_M and in the derivatives of ρ . Furthermore, the $d\nu_{\tilde{u}}$ integral vanishes unless $|\sigma| + |\alpha_2|$ is even. Since $|\alpha_2| \equiv n \pmod{2}$, this implies $|\sigma| + n$ is even so terms involving fractional powers of t vanish as claimed. This leads to exactly the sort of interior expansion described in Theorem 1.1.

4.4. The proof of Theorem 1.1. We now return to the general setting and deal with the case in which the coordinate chart meets the boundary. We set $x = (r, y)$; the $d\nu_r$ integral ranges over $0 \leq r < \infty$ and the $d\nu_y$ integral ranges over $y \in \mathbb{R}^{m-1}$. The dy and $d\tilde{y}$ integrals are handled using the analysis of Lemma 4.3. We therefore suppress these variables and concentrate on the $d\nu_r$ integrals and in essence assume that we are dealing with a 1-dimensional problem; we can always choose the coordinates so $ds^2 = dr^2 + g_{ab}(r, y) dy^i dy^j$. We resume the computation with Equation (4.k) where we do not perform the integrals in the two variables normal to the boundary. We suppress other elements of the notation to examine an integral of the form:

$$f(t) := t^{(n-1)/2} \int_0^\infty \int_0^\infty e^{-\|x-\tilde{x}\|_e^2/(4t)} \left(\frac{X-\tilde{X}}{\sqrt{t}}\right)^{\alpha_2} \langle r_{n,j,\alpha}(x, D_M) \phi(x), \rho(\tilde{x}) \rangle d\nu_x d\nu_{\tilde{x}}.$$

Here the integral has compact support in (x, \tilde{x}) and is homogeneous of degree n in the derivatives of the symbol of D_M , in the derivatives of ϕ , and in the derivatives of ρ . We suppress the role of $|_g$ in the tangential integrals which can also depend on the normal parameter. We may use the binomial theorem to expand $(X - \tilde{X})^{\alpha_2}$ and absorb the relevant powers of x and \tilde{x} into ϕ and ρ ; this alters a and b appropriately and the powers of \sqrt{t} reflect this. We also replace ϕ by $r_{n,j,\alpha} \phi$. We may therefore take $\alpha_2 = 0$ and $r_{n,j,\alpha} = 1$ when establishing the existence of the appropriate asymptotic series.

To simplify the discussion, we may assume that ϕ and ρ have their support contained in $[0, 1]$ and vanish identically near $x = 1$. Let Ξ be a monotonically decreasing smooth cut-off function of compact support which is identically 1 on the support of ϕ and of ρ ; $\Xi(x)$ is identically 1 near $x = 0$ and identically 0 near $x = 1$. We may use Taylor's theorem to express

$$\phi(x) = \sum_{j=0}^N \phi_j x^{j-a} + \Phi \text{ and } \rho(x) = \sum_{j=0}^N \rho_j x^{j-b} + \tilde{\Phi}$$

where Φ and $\tilde{\Phi}$ are C^k and vanish to order k at the origin 0 where k becomes arbitrarily large as $N \rightarrow \infty$. This reduces the problem to considering integrals of the form:

$$\begin{aligned} f_{u,v,\Xi}(t) &= \int_0^1 \int_0^1 e^{-|x-\tilde{x}|^2/(4t)} \Xi(x) \Xi(\tilde{x}) x^{-u} \tilde{x}^{-v} d\tilde{x} dx, \\ f_{u,\tilde{\Phi},\Xi}(t) &:= \int_0^1 \int_0^1 e^{-|x-\tilde{x}|^2/(4t)} \Xi(x) \Xi(\tilde{x}) x^{-u} \tilde{\Phi}(\tilde{x}) d\tilde{x} dx, \\ f_{\Phi,\tilde{\Phi},\Xi}(t) &:= \int_0^1 \int_0^1 e^{-|x-\tilde{x}|^2/(4t)} \Xi(x) \Xi(\tilde{x}) \Phi(x) \tilde{\Phi}(\tilde{x}) d\tilde{x} dx, \end{aligned}$$

where $(u, v) \in \mathcal{O}$, where $\Phi \in L^1$, where $\tilde{\Phi}$ is C^k , where $\tilde{\Phi}$ vanishes near $x = 1$, and where $\tilde{\Phi}$ vanishes to order k at $x = 0$. We use Corollary 3.11 to show that the functions $f_{u,\tilde{\Phi},\Xi}(t)$ have an appropriate

asymptotic series. Extend Φ and $\tilde{\Phi}$ to be zero on $[-1, 0]$; Φ is still in L^1 and $\tilde{\Phi}$ is in C^k . Express

$$\begin{aligned} f_{u, \tilde{\Phi}, \Xi}(t) &= \int_{\mathbb{R}} \int_{\mathbb{R}} e^{-|x-\tilde{x}|^2/(4t)} \Xi(x) \Xi(\tilde{x}) x^{-u} \tilde{\Phi}(\tilde{x}) d\tilde{x} dx, \\ f_{\Phi, \tilde{\Phi}, \Xi}(t) &= \int_{\mathbb{R}} \int_{\mathbb{R}} e^{-|x-\tilde{x}|^2/(4t)} \Xi(x) \Xi(\tilde{x}) \Phi(x) \tilde{\Phi}(\tilde{x}) d\tilde{x} dx \end{aligned}$$

and the desired asymptotic series follows from Lemma 4.3. This completes the proof of Theorem 1.1 \square

4.5. Logarithmic terms if $a + b = 1 - 2k$. Theorem 1.1 has dealt with the case $a + b \neq 1 - 2k$. The following is an immediate consequence of Lemma 3.14 and the arguments we have given above.

Theorem 4.4. *Let $(a, b) \in \mathbb{R}^2$ satisfy $\Re(a) < 1$, $\Re(b) < 1$, and $a + b = 1 - 2k$ where k is a non-negative integer. Let D_M be an operator of Laplace type on a smooth vector bundle V over a compact Riemannian manifold (M, g) without boundary. Let Ω be a compact subdomain of M with smooth boundary. Let $\phi \in C^\infty(V|_{\text{int}(\Omega)})$ and let $\rho \in C^\infty(V^*|_{\text{int}(\Omega)})$. We assume that $r^a \phi$ and $r^b \rho$ are smooth near the boundary of Ω . Let $\beta_\Omega(\phi, \rho, D_M)(t)$ be the heat content of Ω in M . Then there is a complete asymptotic expansion of $\beta_\Omega(\phi, \rho, D_M)(t)$ for small time such that for any positive integer $N > 2k + 3$ so that as $t \downarrow 0$:*

$$\beta_\Omega(\phi, \rho, D_M)(t) = \sum_{n=0}^{2N+2k} t^{n/2} \beta_{n,a,b}(\phi, \rho, D_M) + \sum_{\ell=0}^N t^{\ell+k} \log(t) \tilde{\beta}_{\ell,a,b}(\phi, \rho, D_M) + O(t^{N+k}).$$

The coefficient $\tilde{\beta}_{\ell,a,b}(\phi, \rho, D_M)$ of $t^k \log(t)$ for $\ell = 0$ is given by

$$\tilde{\beta}_{0,a,b}^{\partial\Omega}(\phi, \rho, D_M) = -\frac{a(a+1) \dots (a+2(k+1)-1)}{2 \cdot k!} \int_{\partial\Omega} \langle \phi_0, \rho_0 \rangle dy.$$

More generally, the remaining coefficients are locally computable as suitable regularized integrals over M and integrals over the boundary.

5. HEAT CONTENT ASYMPTOTICS WITH NEUMANN AND DIRICHLET BOUNDARY CONDITIONS

Let M be a compact smooth m dimensional manifold with smooth boundary ∂M . Let $e^{-t\Delta_\pm}$ be the fundamental solution of the heat equation for the Neumann (+) or the Dirichlet (−) realization of the Laplace-Beltrami operator. If $\phi, \rho \in L^1(M)$, then we may define:

$$\beta_\pm^M(\phi, \rho)(t) := \int_M e^{-t\Delta_\pm} \phi \cdot \rho dx.$$

In a subsequent paper, we plan to investigate the general setting; this will involve extending the analysis of Section 4 to examine elliptic boundary conditions. For the moment, however, in the interests of brevity, we content ourselves with establishing an improved version of Conjecture 1.2 of [?] in the 1-dimensional setting where $M = [0, 1]$ and where $\Delta = -\partial_x^2$:

Theorem 5.1. *Let $\Re(a) < 1$ and $\Re(b) < 1$. Let ϕ and ρ be smooth on $(0, 1)$. Assume $r^a \phi$ and $r^b \rho$ are smooth near the boundary of $M = [0, 1]$. Let $\Delta = -\partial_x^2$.*

- (1) *If $a + b \neq 1, -1, -3, \dots$, then there is a complete asymptotic series as $t \downarrow 0$ with locally computable coefficients of the form:*

$$\beta_\pm^{[0,1]}(\phi, \rho)(t) \sim \sum_{n=0}^{\infty} t^n \beta_{\pm,n}(\phi, \rho) + \sum_{j=0}^{\infty} t^{(1+j-a-b)/2} \beta_{\pm,j,a,b}(\phi, \rho).$$

- (2) *If a is real and if $a + b = -2k - 1$ is an odd integer, then there is a complete asymptotic series as $t \downarrow 0$ with locally computable coefficients of the form:*

$$\beta_\pm^{[0,1]}(\phi, \rho)(t) \sim \sum_{n=0}^{\infty} t^{n/2} \beta_{\pm,n,a,b}(\phi, \rho) + \sum_{\ell=0}^{\infty} t^{\ell+k} \log(t) \tilde{\beta}_{\pm,\ell,a,b}(\phi, \rho).$$

Remark 5.2. We can take Dirichlet boundary conditions at one end and Neumann boundary conditions at the other end of the interval and obtain appropriate asymptotic series. We can also let the growth and decay rates differ at the two components and again obtain appropriate asymptotic series.

5.1. The half line. Before proving Theorem 5.1, we must examine the heat content asymptotics on the half-line. Let $\phi \in L^1([0, \infty))$. Set

$$T\phi(x; t) := \frac{1}{\sqrt{4\pi t}} \int_0^\infty e^{-(x+\tilde{x})^2/(4t)} \phi(\tilde{x}) d\tilde{x}, \quad \text{and}$$

$$H_\pm(\phi)(x; t) := \frac{1}{\sqrt{4\pi t}} \int_0^\infty \left\{ e^{-(x-\tilde{x})^2/(4t)} \pm e^{-(x+\tilde{x})^2/(4t)} \right\} \phi(\tilde{x}) d\tilde{x}.$$

Let $e^{-t\Delta_+}$ and $e^{-t\Delta_-}$ be the Neumann (+) and Dirichlet realizations (−) of the Laplacian $\Delta = -\partial_x^2$ on the half line. The following result is well known. We shall give the proof in the interests of completeness and to establish notation.

Lemma 5.3. Let $\phi \in L^1([0, \infty))$.

- (1) $\|T\phi(\cdot; t)\|_{L^1} \leq \|\phi\|_{L^1}$.
- (2) If $\delta > 0$, then $\|T(\chi_{[\delta, \infty)}\phi)(\cdot; t)\|_{L^1} \leq 2e^{-\delta^2/(8t)} \|\phi\|_{L^1}$.
- (3) $\lim_{t \downarrow 0} \|T\phi(\cdot; t)\|_{L^1} = 0$.
- (4) $(\partial_t + \Delta)T\phi = 0$.
- (5) $\{e^{-t\Delta_\pm}\phi\}(x; t) = H_\pm(\phi)(x; t)$

Proof. We establish Assertion (1) by estimating:

$$\begin{aligned} \|T\phi\|_{L^1} &\leq \frac{1}{\sqrt{4\pi t}} \int_0^\infty \int_0^\infty e^{-(x+\tilde{x})^2/(4t)} |\phi(\tilde{x})| d\tilde{x} dx \leq \frac{1}{\sqrt{4\pi t}} \int_0^\infty \int_0^\infty e^{-x^2/(4t)} |\phi(\tilde{x})| d\tilde{x} dx \\ &= \frac{1}{\sqrt{4\pi t}} \int_0^\infty e^{-x^2/(4t)} dx \cdot \int_0^\infty |\phi(\tilde{x})| d\tilde{x} \leq \|\phi\|_{L^1}. \end{aligned}$$

We prove Assertion (2) by computing similarly that:

$$\begin{aligned} \|T(\chi_{[\delta, \infty)}\phi)(\cdot; t)\|_{L^1} &\leq \frac{1}{\sqrt{4\pi t}} \int_0^\infty \int_\delta^\infty e^{-(x+\tilde{x})^2/(4t)} |\phi(\tilde{x})| d\tilde{x} dx \\ &\leq \frac{1}{\sqrt{4\pi t}} \int_0^\infty \int_\delta^\infty e^{-(x+\delta)^2/(4t)} |\phi(\tilde{x})| d\tilde{x} dx \\ &\leq \frac{1}{\sqrt{4\pi t}} e^{-\delta^2/(8t)} \int_0^\infty e^{-x^2/(8t)} dx \cdot \int_0^\infty |\phi(\tilde{x})| d\tilde{x} \leq 2e^{-\delta^2/(8t)} \|\phi\|_{L^1}. \end{aligned}$$

Let $\epsilon > 0$ be given. Choose $\delta > 0$ so $\|\chi_{[0, \delta]}\phi\|_{L^1} < \frac{\epsilon}{2}$. Choose t_0 so $0 < t < t_0$ implies $2e^{-\delta^2/(8t)} \|\phi\|_{L^1} < \frac{\epsilon}{2}$. Assertion (3) now follows from Assertion (4). To show that $(\partial_t + \Delta)T\phi = 0$, we compute:

$$\begin{aligned} \partial_t T\phi(x; t) &= \frac{1}{\sqrt{4\pi t}} \int_0^\infty \left\{ -\frac{1}{2t} + \frac{(x+\tilde{x})^2}{4t^2} \right\} e^{-(x+\tilde{x})^2/(4t)} \phi(\tilde{x}) d\tilde{x}, \\ \partial_x^2 T\phi(x; t) &= \frac{1}{\sqrt{4\pi t}} \int_0^\infty \left\{ -\frac{2}{4t} + \frac{4(x+\tilde{x})^2}{(4t)^2} \right\} e^{-(x+\tilde{x})^2/(4t)} \phi(\tilde{x}) d\tilde{x}, \\ (\partial_t + \Delta)T\phi(x; t) &= (\partial_t - \partial_x^2)H(x; t) = 0. \end{aligned}$$

We use Assertion (3) and Assertion (4) to see $(\partial_t + \Delta)H_\pm(\phi)(x; t) = 0$ and $\lim_{t \downarrow 0} H_\pm(\phi)(\cdot; t) = \phi(\cdot)$ in L^1 . We complete the proof of the Lemma by checking that the boundary conditions are satisfied:

$$\begin{aligned} \{e^{-t\Delta_-}\phi\}(0; t) &= \frac{1}{\sqrt{4\pi t}} \int_0^\infty \left\{ e^{-(0-\tilde{x})^2/(4t)} - e^{-(0+\tilde{x})^2/(4t)} \right\} \phi(\tilde{x}) d\tilde{x} = 0, \\ \partial_x \{e^{-t\Delta_+}\phi\}(0; t) &= \frac{1}{\sqrt{4\pi t}} \int_0^\infty \frac{2}{4t} \left\{ \tilde{x}e^{-(0-\tilde{x})^2/(4t)} - \tilde{x}e^{-(0+\tilde{x})^2/(4t)} \right\} \phi(\tilde{x}) d\tilde{x} = 0. \end{aligned} \quad \square$$

5.2. The proof of Theorem 5.1. The critical case to examine is $\phi(x) = x^{-a}$ and $\rho(x) = x^{-b}$; the remainder of the analysis of the general case then follows using exactly the same arguments using cut-off functions as was done previously. And thus all that is necessary to do is to examine the correction term

$$\tilde{\beta}(a, b; t) := \frac{1}{\sqrt{4\pi t}} \int_0^1 \int_0^1 e^{-(x+\tilde{x})^2/(4t)} x^{-a} \tilde{x}^{-b} dx d\tilde{x}$$

as then Theorem 5.1 will follow from our previous results. We can replace $[0, 1]$ by $[0, \infty)$ modulo an exponentially suppressed error term as $t \downarrow 0$. This expresses

$$\begin{aligned} \tilde{\beta}(a, b; t) &\sim \frac{1}{\sqrt{4\pi t}} \int_0^\infty \int_0^\infty e^{-(x+\tilde{x})^2/(4t)} x^{-a} \tilde{x}^{-b} dx d\tilde{x} \\ &= t^{(1-a-b)/2} \int_0^\infty \int_0^\infty e^{-(x+\tilde{x})^2} x^{-a} \tilde{x}^{-b} dx d\tilde{x}. \end{aligned}$$

This integral can be evaluated using the techniques in [7] (see the proof of Lemma 1.6) to yield the following formula for the correction term:

$$\tilde{\beta}(a, b; t) \sim t^{(1-a-b)/2} \cdot 2^{-a-b} \pi^{-1/2} \Gamma\left(\frac{2-a-b}{2}\right) \cdot \frac{\Gamma(1-a)\Gamma(1-b)}{\Gamma(2-a-b)}. \quad \square$$

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